Finite Automata

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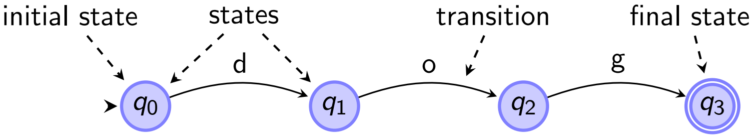
January 20, 2023

Processing Regular Expressions

- We already learned about Java's regular expression functionality
- Now we get to know the machinery behind
  - `Pattern` and
  - `Matcher` classes
- Compiling a regular expression into a `Pattern` object produces a Finite Automaton
- This automaton is then used to perform the matching tasks
- We will see how to construct a finite automaton that recognizes an input string, i.e., tries to find a full match

Definition: Finite Automaton

- A finite automaton (FA) is a tuple $A =< Q, \Sigma, \delta, q_0, F >$
  - Q a finite non-empty set of states
  - Σ a finite alphabet of input letters
  - δ a (total) transition function $Q \times \Sigma \longrightarrow Q$
  - $q_0 \in Q$ the initial state
  - $F \subseteq Q$ the set of final (accepting) states
- Transition graphs (diagrams):



Finite Automata: Matching

- A finite automaton accepts a given input string s if there is a sequence of states $p_1, p_2, \ldots, p_{|s|} \in Q$ such that
  1. $p_1 = q_0$, the start state
  2. $\delta(p_i, s_i) = p_{i+1}$, where s_i is the i-th character in s
  3. $p_{|s|} \in F$, i.e., a final state
- A string is successfully matched if we have found the appropriate sequence of states
- Imagine the string on an input tape with a pointer that is advanced when using a δ transition
- The set of strings accepted by an automaton is the accepted language, analogous to regular expressions

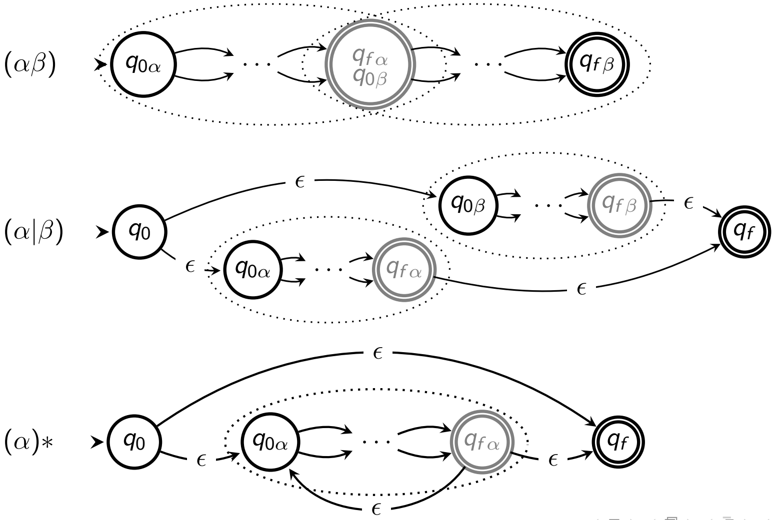
(Non)deterministic Automata

- in the definition of automata, δ was a total function $\Rightarrow$

  given an input string, the path through the automaton is uniquely determined
- those automata are therefore called deterministic
- for nondeterministic FA, δ is a transition relation
- $\delta : Q \times \Sigma \cup \{\epsilon\} \longrightarrow \mathcal{P}(Q)$, where $\mathcal{P}(Q)$ is the powerset of Q
- allows transitions from one state into several states with the same input symbol
- need not be total
- can have transitions labeled ϵ (not in Σ), which represents the empty string

RegExps $\longrightarrow$ Automata

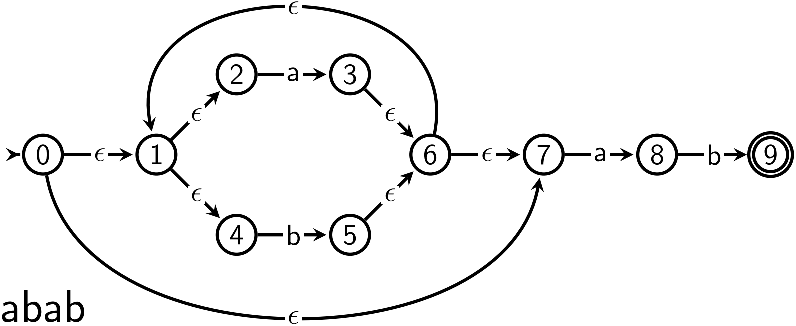
Construct nondeterminstic automata from regular expressions



NFA vs. DFA

- Traversing a DFA is easy given the input string: the path is uniquely determined
- In contrast, traversing an NFA requires keeping track of a set of (current) states, starting with the set $\{q_o\}$
- Processing the next input symbol means taking all possible outgoing transitions from this set and collecting the new set
- From every NFA, an equivalent DFA (one which does accept the same language), can be computed
- Basic Idea: track the subsets that can be reached for every possible input

Traversing an NFA



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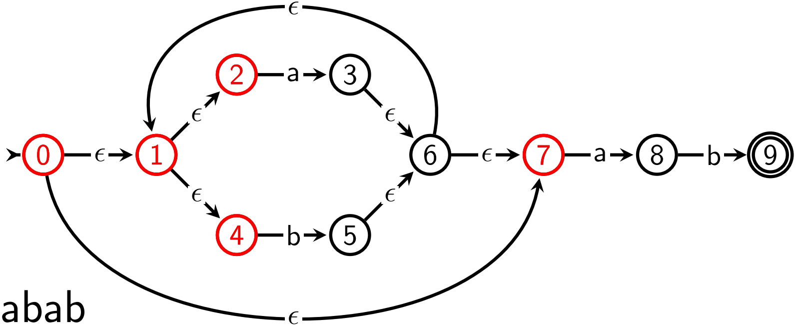
Traversing an NFA



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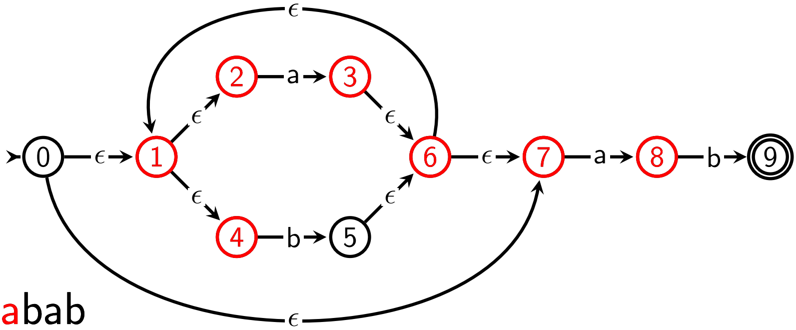

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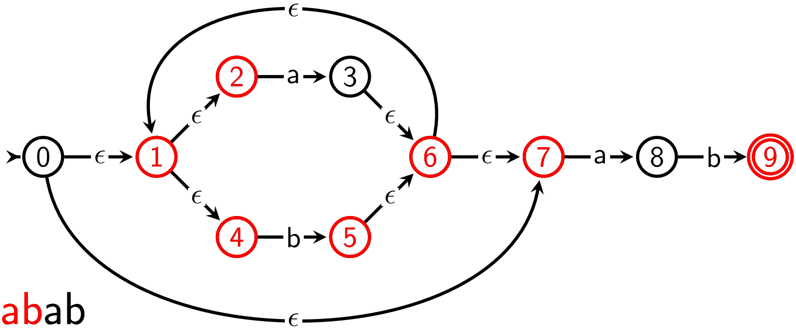

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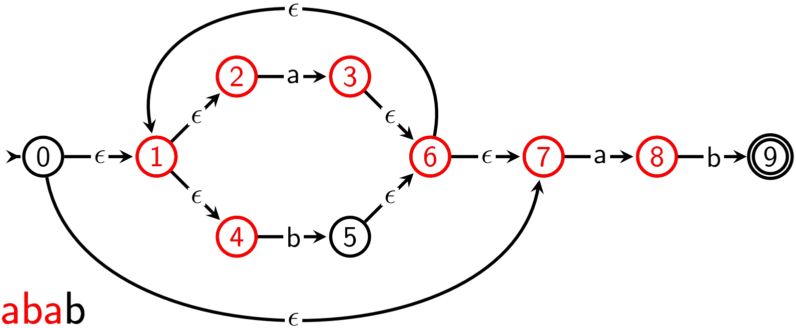

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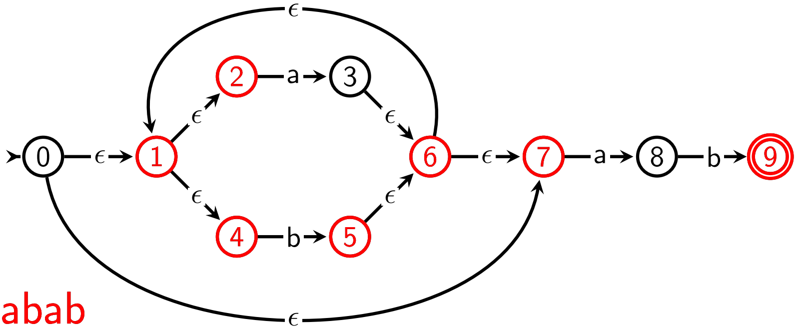

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NFA $\longrightarrow$ DFA: Subset Construction

- Simulate “in parallel” all possible moves the automaton can make
- The states of the resulting DFA will represent sets of states of the NFA, i.e., elements of $\mathcal{P}(Q)$
- We use two operations on states/state-sets of the NFA

ϵ-*closure*(T)	Set of states reachable from any state s in T on ϵ-transitions
move(T, a)	Set of states to which there is a transition from one state in T on input symbol a

- The final states of the DFA are those where the corresponding NFA subset contains a final state

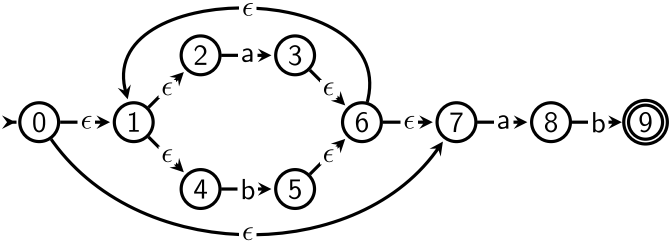
Algorithm: Subset Construction

```
DFAStates = $\epsilon$-closure($\{s_0\}$)
while there is an unmarked state $T$ in DFAStates do
    mark $T$
    for each input symbol $a$ do
        $U$ := $\epsilon$-closure(move($T$, $a$))
        DFADelta[$T$, $a$] := $U$
        if $U \notin$ DFAStates then
            add $U$ as unmarked to DFAStates
```

ϵ-closure(T):

```
$\epsilon$-closure := $T$; to_check := $T$
while to_check not empty do
    get some state $t$ from to_check
    for each state $u$ with edge labeled $\epsilon$ from $t$ to $u$ do
        if $u \notin \epsilon$-closure then
            add $u$ to $\epsilon$-closure and to_check
```

Example: Subset construction



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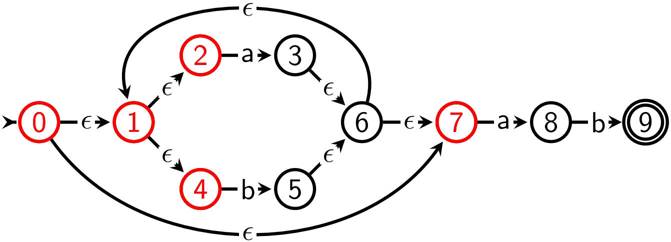

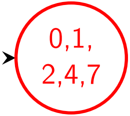

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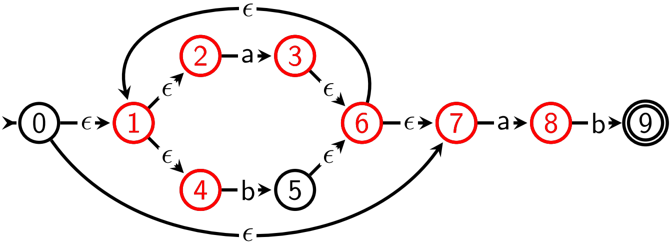

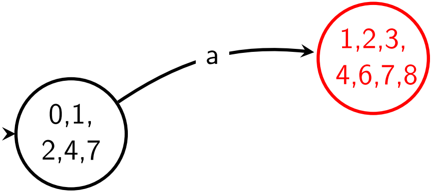

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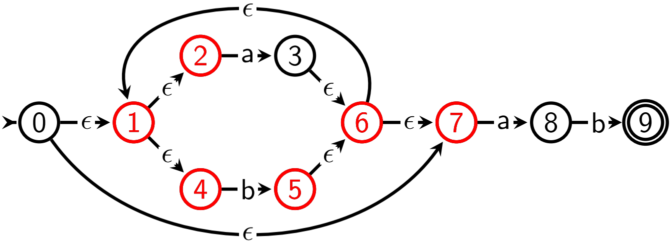

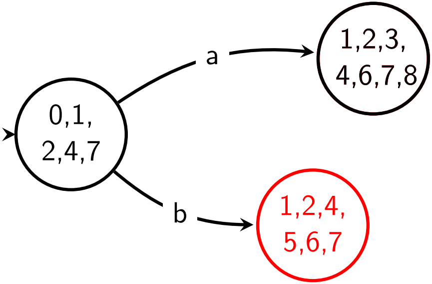

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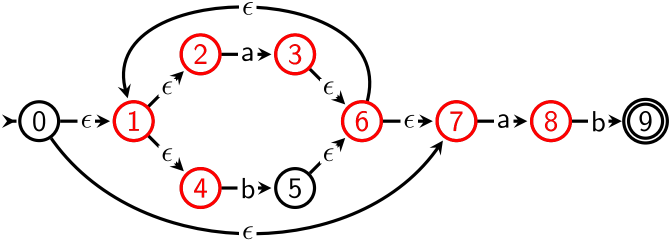

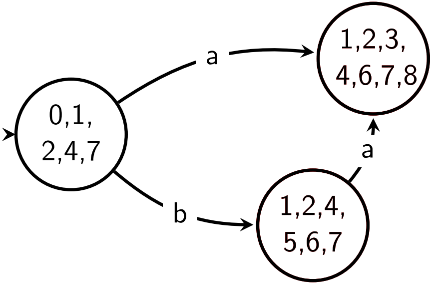

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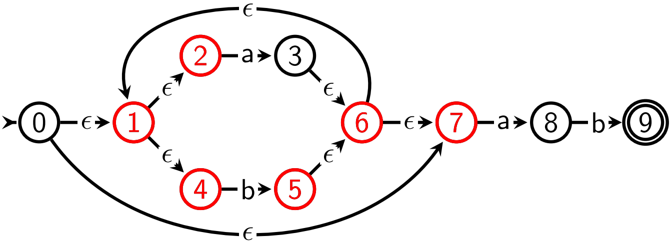

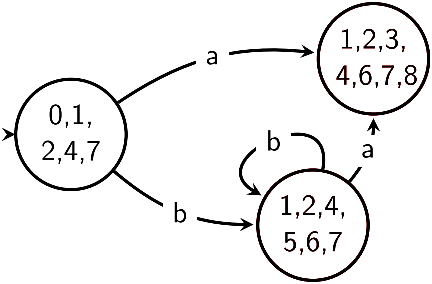

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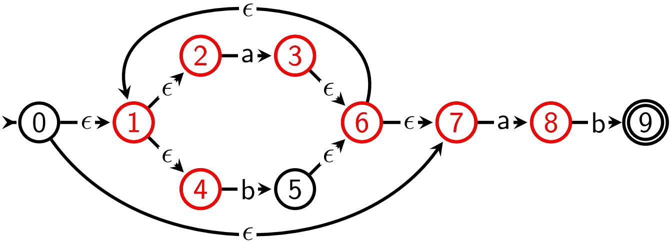

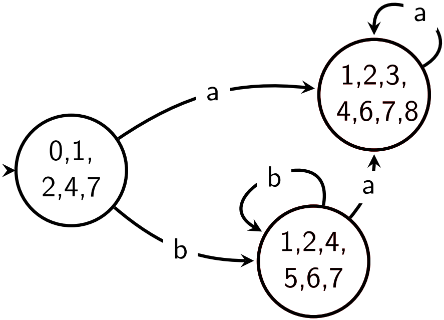

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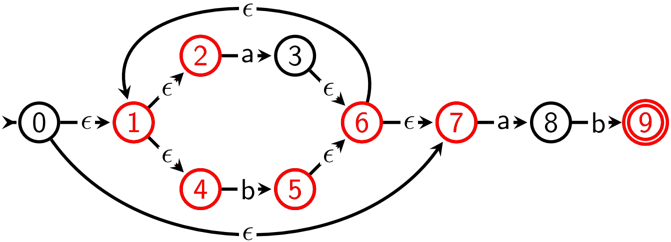

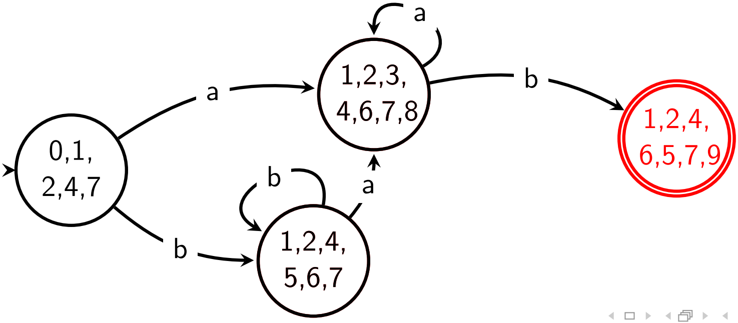

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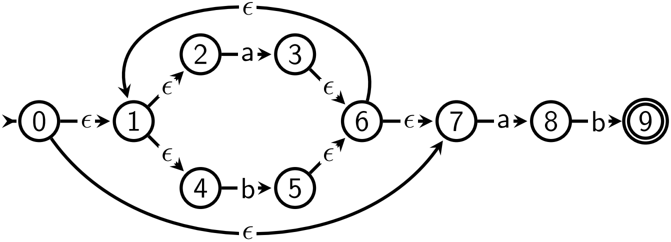

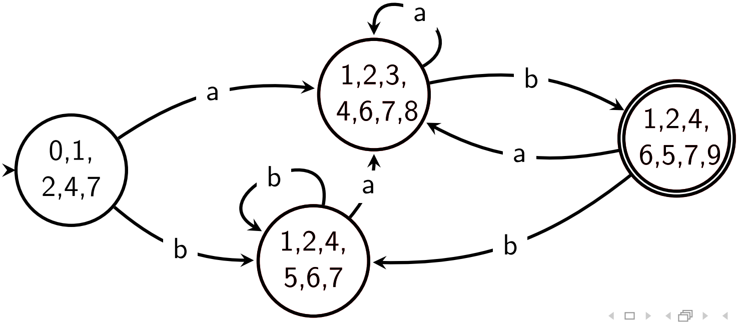

Time/Space Considerations

- DFA traversal is linear to the length of input string x
- NFA needs $\mathcal{O}(n)$ space (states+transitions), where n is the length of the regular expression
- NFA traversal may need time $n \times |x|$, so why use NFAs?

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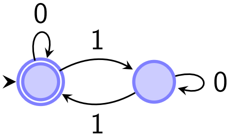
- NFA traversal may need time $n \times |x|$, so why use NFAs?
- There are DFA that have at least 2^n states!
- Solution 1: "Lazy" construction of the DFA: construct DFA states on the fly up to a certain amount and cache them
- Solution 2: Try to minimize the DFA:
  There is a unique (modulo state names) minimal automaton for a regular language!

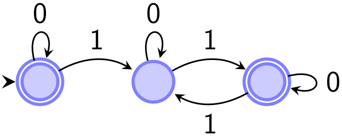
Minimal Automata

- Take any state q of the deterministic automaton to minimize and assume it to be the (single) start state
- We call the language that this automaton accepts the right language of q
- The language of each state consists of suffixes of the overall accepted language
- If two states accept the same language, they are equivalent and can be merged
- To minimize the automaton, merge all equivalent nodes
- This is implemented by first partitioning the original set of states into equivalence classes, i.e., sets of equivalent states
- Finally, each equivalence class is replaced by a single state, merging transitions accordingly

DFA Minization

- Every DFA defines a unique language
- In general, there may be many DFAs for a given language
- The following DFAs accept the same language





Indistinguishable States

- Two states p and q are called **indistinguishable**, if **for all** $w \in \Sigma^*$, F being the set of final states

  $\delta^*(p, w) \in F \Leftrightarrow \delta^*(q, w) \in F$, and
  $\delta^*(p, w) \notin F \Leftrightarrow \delta^*(q, w) \notin F$
- $\mathcal{L}_p = \{w \in \Sigma^* | \delta^*(p, w) \in F\}$ is also called suffix language of the automaton.
- The states p and q behave in the same way for all possible strings: $\mathcal{L}_p = \mathcal{L}_q$
- When is a state p distinguishable from q?

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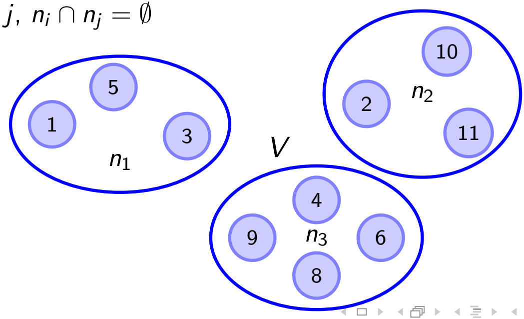
- When is a state p distinguishable from q?
- There is a string w where $\delta^*(p, w) \in F$ and $\delta^*(q, w) \notin F$ or vice versa
- Algorithm: start with strings of length 0 and work yourself backwards through the automaton

Indistinguishable States

- Indistinguishable states behave in an identical way
- As a consequence, a set of indistinguishable states can be merged into a single state
- Indistinguishability is an equivalence relation:
  - Reflexive: Each state is indistinguishable from itself
  - Symmetric: If p is indistinguishable from q, then q is indistinguishable from p
  - Transitive: If p is indistinguishable from q, and q is indistinguishable from r , then p is indistinguishable from r .

Indistinguishability And Partitions

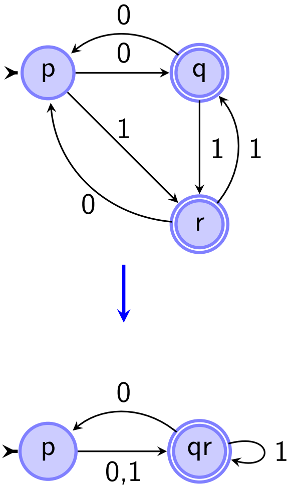
- Indistinguishability is an equivalence relation:
  - Reflexive: Each state is indistinguishable from itself
  - Symmetric: If p is indistinguishable from q, then q is indistinguishable from p
  - Transitive: If p is indistinguishable from q, and q is indistinguishable from r, then p is indistinguishable from r.
- (Total) equivalence relations on nodes V of a graph induce a (total) partitioning into subsets of nodes $\mathcal{N} = n_1, n_2, \ldots, n_k$, such that
  - For all i and j, $n_i \cap n_j = \emptyset$
  - $\bigcup_i n_i = V$



Detecting Indistinguishable States

- Basis: Every nonaccepting state is distinguishable from any accepting state ($w = \epsilon$)
- Induction: States p and q are distinguishable if there is some input symbol a such that $\delta(p, a)$ is distinguishable from $\delta(q, a)$
- Testing all possible state pairs until no more distinguishable states can be found leaves the rest indistinguishable
- The sets of indistinguishable states can be merged into one state per set

Detecting Indistinguishable States



- Basis: p distinguishable from q and r
- 0: q and r go to p, does not separate them
- 1: q goes to r, and r to q, which are indistinguishable, again, no separation
- q and r can be merged into a single state

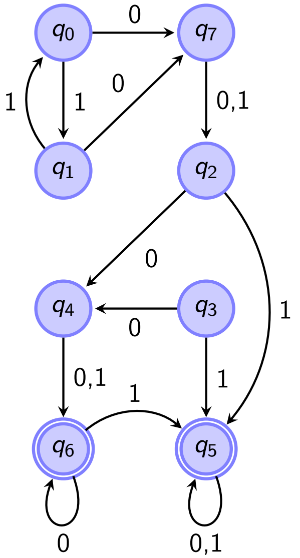
DFA Minimization: the Algorithm

Create an $n \times n$ boolean table distinct, n the number of states, and set all cells to *false*

```
**for** every pair $(p, q), p, q \in Q$ **do**
    **if** $p \in F \wedge q \notin F$ or vice versa **then**
        $\texttt{distinct}(p, q) = \text{true}$
**while** there is a change in distinct **do**
    **for** every pair $(p, q), p, q \in Q$ and each $a \in \Sigma$ **do**
        **if** $\neg\texttt{distinct}(p, q) \wedge \texttt{distinct}(\delta(p, a), \delta(q, a))$ **then**
            $\texttt{distinct}(p, q) = \text{true}$
**for** $p \in Q$ **do**          // assume an order on the states
    **for** $q > p \in Q$ **do**
        **if** $\neg\texttt{distinct}(p, q)$ **then**
            merge $q$ into $p$, mark $q$ deleted
```

Minimization in action

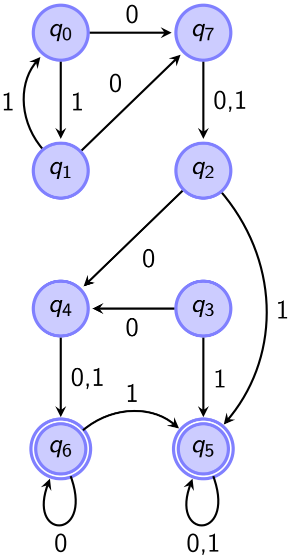
	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7
q_7						ϵ	ϵ	
q_6	ϵ	ϵ	ϵ	ϵ	ϵ			
q_5	ϵ	ϵ	ϵ	ϵ	ϵ			
q_4								
q_3								
q_2								
q_1								
q_0								



Minimization in action

	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7
q_7						ϵ	ϵ	
q_6	ϵ	ϵ	ϵ	ϵ	ϵ			
q_5	ϵ	ϵ	ϵ	ϵ	ϵ			
q_4								
q_3								
q_2	1							
q_1								
q_0								

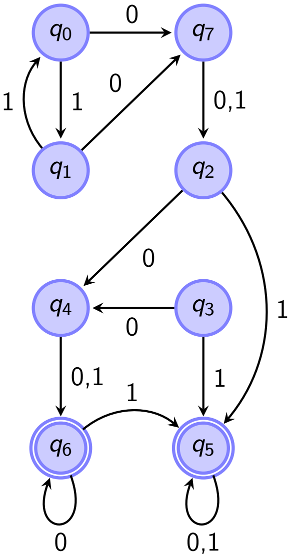
Iteration 1



Minimization in action

	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7
q_7						ϵ	ϵ	
q_6	ϵ	ϵ	ϵ	ϵ	ϵ			
q_5	ϵ	ϵ	ϵ	ϵ	ϵ			
q_4								
q_3	1							
q_2	1							
q_1								
q_0								

Iteration 1



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q_6	ϵ	ϵ	ϵ	ϵ	ϵ			
q_5	ϵ	ϵ	ϵ	ϵ	ϵ			
q_4	0							
q_3	1							
q_2	1							
q_1								
q_0								

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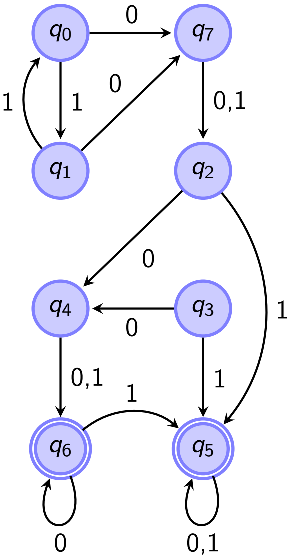

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	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7
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q_6	ϵ	ϵ	ϵ	ϵ	ϵ			
q_5	ϵ	ϵ	ϵ	ϵ	ϵ			
q_4	0							
q_3	1							
q_2	1	1						
q_1								
q_0								

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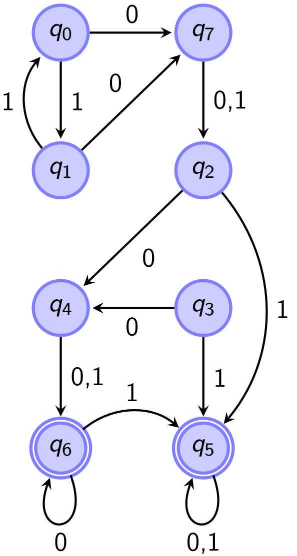

Minimization in action

	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7
q_7						ϵ	ϵ	
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q_4	0							
q_3	1	1						
q_2	1	1						
q_1								
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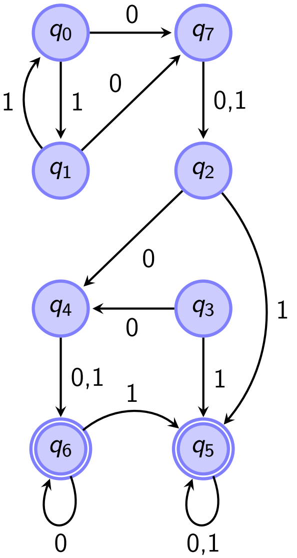

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q_6	ϵ	ϵ	ϵ	ϵ	ϵ			
q_5	ϵ	ϵ	ϵ	ϵ	ϵ			
q_4	0	0						
q_3	1	1						
q_2	1	1						
q_1								
q_0								

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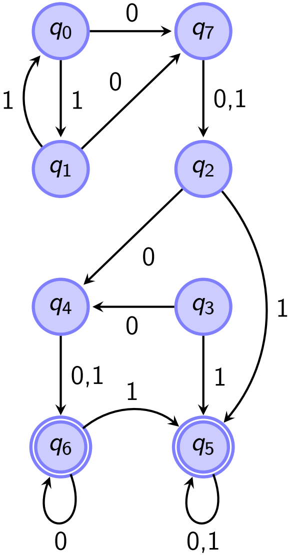

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q_6	ϵ	ϵ	ϵ	ϵ	ϵ			
q_5	ϵ	ϵ	ϵ	ϵ	ϵ			
q_4	0	0						
q_3	1	1						
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q_1								
q_0								

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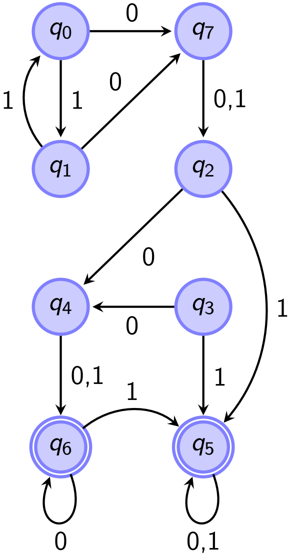

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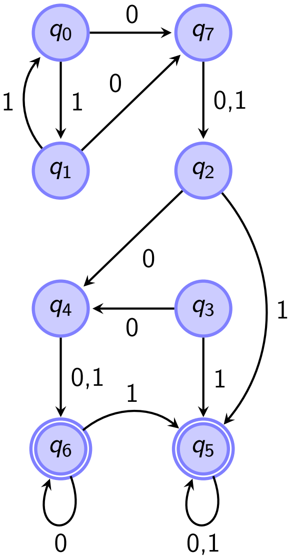
	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7
q_7		1				ϵ	ϵ	
q_6	ϵ	ϵ	ϵ	ϵ	ϵ			
q_5	ϵ	ϵ	ϵ	ϵ	ϵ			
q_4	0	0	0					
q_3	1	1						
q_2	1	1						
q_1								
q_0								

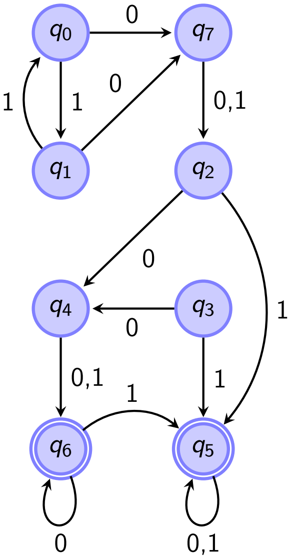
Iteration 1



Minimization in action

	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7
q_7		1	0			ϵ	ϵ	
q_6	ϵ	ϵ	ϵ	ϵ	ϵ			
q_5	ϵ	ϵ	ϵ	ϵ	ϵ			
q_4	0	0	0					
q_3	1	1						
q_2	1	1						
q_1								
q_0								

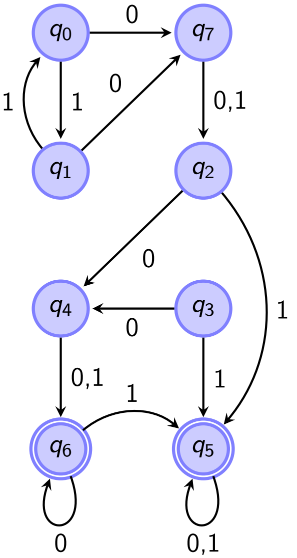
Iteration 1



Minimization in action

	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7
q_7		1	0			ϵ	ϵ	
q_6	ϵ	ϵ	ϵ	ϵ	ϵ			
q_5	ϵ	ϵ	ϵ	ϵ	ϵ			
q_4	0	0	0	0				
q_3	1	1						
q_2	1	1						
q_1								
q_0								

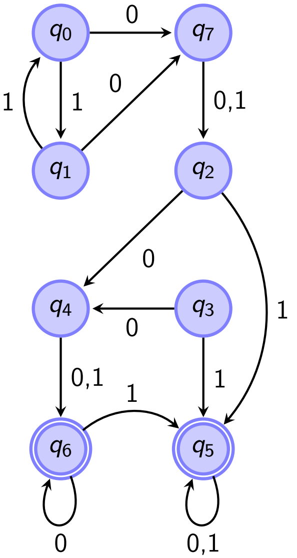
Iteration 1



Minimization in action

	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7
q_7		1	0	0		ϵ	ϵ	
q_6	ϵ	ϵ	ϵ	ϵ	ϵ			
q_5	ϵ	ϵ	ϵ	ϵ	ϵ			
q_4	0	0	0	0				
q_3	1	1						
q_2	1	1						
q_1								
q_0								

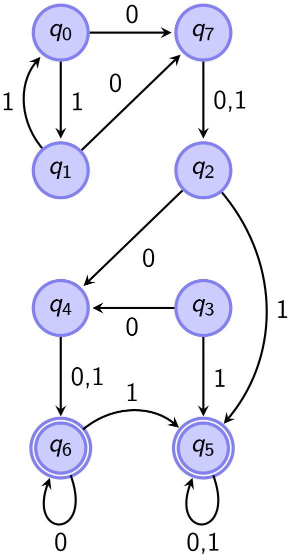
Iteration 1



Minimization in action

	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7
q_7		1	0	0	0	ϵ	ϵ	
q_6	ϵ	ϵ	ϵ	ϵ	ϵ			
q_5	ϵ	ϵ	ϵ	ϵ	ϵ			
q_4	0	0	0	0				
q_3	1	1						
q_2	1	1						
q_1								
q_0								

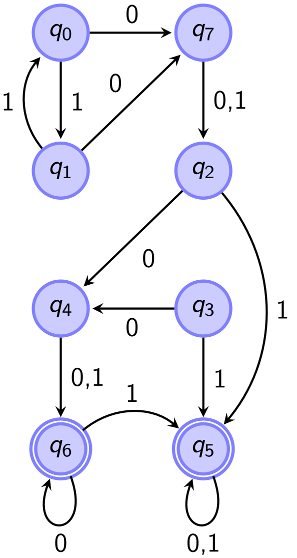
Iteration 1



Minimization in action

	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7
q_7	0	1	0	0	0	ϵ	ϵ	
q_6	ϵ	ϵ	ϵ	ϵ	ϵ			
q_5	ϵ	ϵ	ϵ	ϵ	ϵ			
q_4	0	0	0	0				
q_3	1	1						
q_2	1	1						
q_1								
q_0								

Iteration 2



Minimization in action

	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7
q_7	0	1	0	0	0	ϵ	ϵ	
q_6	ϵ	ϵ	ϵ	ϵ	ϵ			
q_5	ϵ	ϵ	ϵ	ϵ	ϵ			
q_4	0	0	0	0				
q_3	1	1						
q_2	1	1						
q_1								
q_0