Graphs and Search

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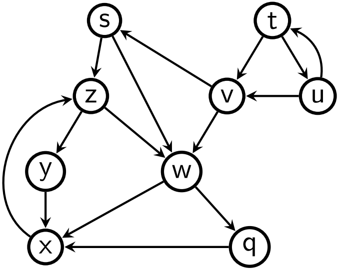
Graphs: Definition

- Graph $\mathcal{G}$: A set of vertices (nodes) $\mathcal{V}$ and a set of edges $\mathcal{E}$, which is a relation on vertices, that is: $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$
- Example:
  - Vertices: students at the university
  - $(u, v) \in \mathcal{E} \Leftrightarrow$ student u knows student v
- Graphical representation:
  - vertices: blobs
  - edges: arrows (arcs) between the blobs
- If $\mathcal{E}$ is symmetric, i.e., if $(u, v) \in \mathcal{E} \Leftrightarrow (v, u) \in \mathcal{E}$ the graph is called undirected (plain arcs, not arrows)
- Example: $\mathcal{E}$ is the set of students that are akin

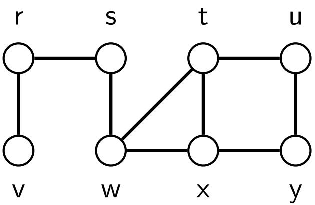
Graphs: Definitions II

- Vertex u is reachable from vertex v ($u \rightarrow v$) iff there is a sequence of edges $(u, w_1), (w_1, w_2), \ldots, (w_n, v)$ in $\mathcal{E}$
- A graph is cyclic (contains a cycle) if there is $u \in \mathcal{V}$ s.th. $u \rightarrow u$ over a nontrivial sequence of edges in $\mathcal{E}$, including a self loop

directed graph



undirected graph



Implementation Basics

- Represent the vertices as numbers from zero to $|\mathcal{V}| - 1$
- Matrix representation: represent $\mathcal{E}$ as a quadratic `boolean` matrix A of size $|\mathcal{V}|$; $A[i, j]$ is `true` iff $(i, j) \in \mathcal{E}$

\+ Good for dense graphs, where $|\mathcal{E}| \approx |\mathcal{V}|^2$: only one bit per edge

\+ Fast: are two vertices directly connected?

– Initialization is quadratic in $|\mathcal{V}|$

– Visiting all outgoing edges of a vertex takes $|\mathcal{V}|$ steps, no matter how many there really are

– Additional information attached to the edges (e.g., weights) has to be stored separately

Adjacency List Representation

- For every vertex, store a list of outgoing edges, i.e., the vertex number that is reached
- Graph is represented by an array of list heads
- In Java: `ArrayList` of `Lists`.
- \+ Compact representation for most graphs, except if they are very dense
- \+ Allows more efficient implementations of many graph algorithms
- \+ Additional edge information can be stored in the elements of the edge lists directly
- In the exercises, a `Map<Integer, List<Edge>>` is used for simplicity

Search in Graphs

- Task: visit all reachable vertices, starting at vertex s
- Iteratively use all the outgoing edges of s, and all the nodes that can be reached through these egdes
- Make sure that no node gets explored twice
- Basic idea: maintain two sets
  - $\mathcal{U}$ the *visited* nodes
  - $\mathcal{A}$ the *active* nodes, i.e., still unexplored outedges
- In textbooks, vertices are often assigned colors during the search:
  - White: not in $\mathcal{U}$ and not in $\mathcal{A}$
  - Grey: in $\mathcal{U}$ and in $\mathcal{A}$ (under consideration)
  - Black: in $\mathcal{U}$, but not in $\mathcal{A}$ anymore (finished)

Generic Search Algorithm

Initialization: both sets contain only the start vertex s

$\mathcal{U} = \mathcal{A} = \{s\}$ // s gets grey
while $\mathcal{A} \neq \emptyset$ **do**
  for some node $n \in \mathcal{A}$ **do**
    if there is an unused edge $e = (n, m)$ leaving n **then**
      if $m \notin \mathcal{U}$ **then**
        $\mathcal{U} = \mathcal{U} \cup \{m\}; \mathcal{A} = \mathcal{A} \cup \{m\}$ // m gets grey
    else
      $\mathcal{A} = \mathcal{A} - \{n\}$ // n gets black

Questions:

- How to implement sets $\mathcal{U}$ and $\mathcal{A}$?
- Does the result depend on the implementation?

Implementation of $\mathcal{U}$

- What is the best data structure for $\mathcal{U}$?
- What are the operations on $\mathcal{U}$?

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- What are the operations on $\mathcal{U}$?
  1. Add a node m
  2. Is node n contained in the set?
- $\mathcal{U}$ should be implemented as a bit vector over the nodes
- Two alternatives:
  - `boolean` member variable of the node data structure
  - A so-called property vector (or property map) attached to the vertices

Property Vectors

Advantages and drawbacks of property vectors

- More flexible:
  - Create all and only those you need for an algorithm
  - In a graph framework, one can not put all the data into the vertices
  - May contain any type, small or bigger datastructures
  - Only use memory when they are needed
- Require an efficient indexing between vertices and values: maintain a numeric index in the vertices
- Member variables are always faster

Property vectors can also be used for graph edges

Implementation of $\mathcal{A}$

The choice of the data structure for $\mathcal{A}$ and the decisions about n and e determine the order in which vertices are visited

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Implementation of $\mathcal{A}$

The choice of the data structure for $\mathcal{A}$ and the decisions about n and e determine the order in which vertices are visited

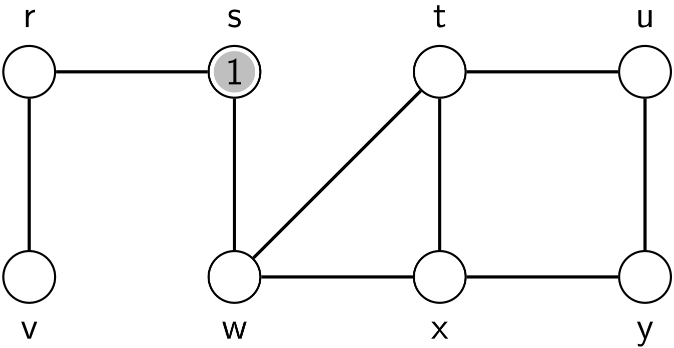
- Operations on set $\mathcal{A}$:
  - Add a vertex
  - Get and remove some vertex (nondeterministic)
  - Test if set is empty
- Implement $\mathcal{A}$ as a queue and keep n until it gets black: Breadth First Search (BFS)
- Implement $\mathcal{A}$ as a stack and always take its top element: Depth First Search (DFS)
- DFS is often implemented as a recursive function, the function call stack takes the role of $\mathcal{A}$

BFS Implementation

```
for all v ∈ 𝒱 do
    d(v) = 0
time = 1          // the time when a vertex is touched
for all v ∈ 𝒱 with d(v) == 0 do        // v is the start node
    d(v) = time; 𝒜.push_back(v)         // v gets grey
    while ¬ 𝒜.empty() do
        n = 𝒜.pop_front()
        time = d(n) + 1
        for all e = (n, m) do
            if d(m) == 0 then          // m ∉ 𝒰 ?
                d(m) = time; 𝒜.push_back(m)        // m gets grey
        // n gets black
```

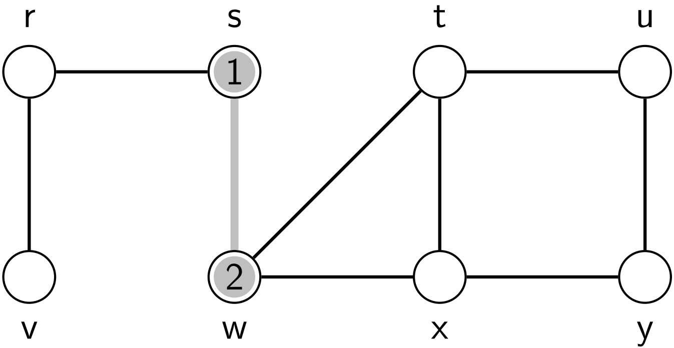
- Finally, all vertices of $\mathcal{G}$ have been visited
- The $d(v)$ is abused to serve as the $\mathcal{U}$ bitvector

Breadth First Search



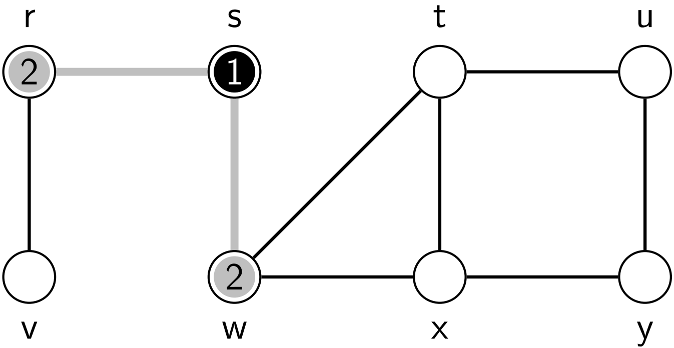
$\mathcal{A} \rightarrow$

Breadth First Search



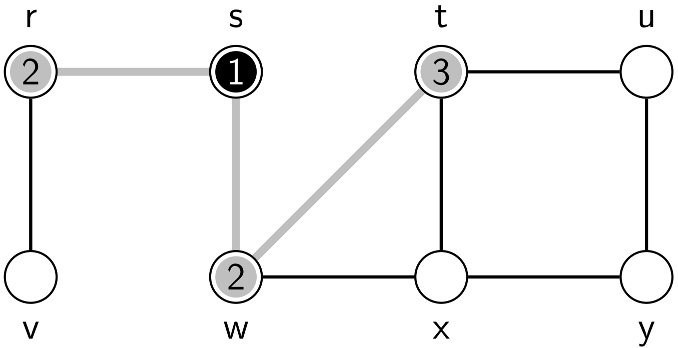
$\mathcal{A} \rightarrow$ **w**

Breadth First Search



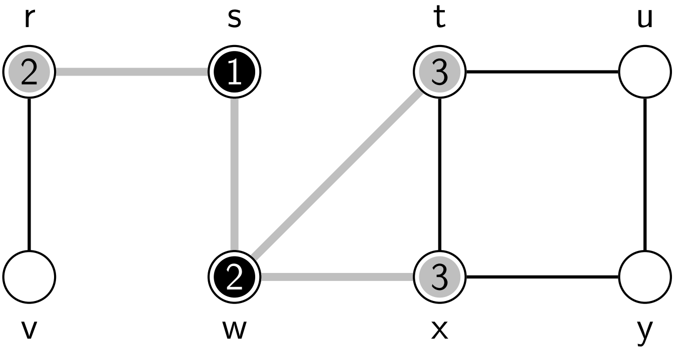
$\mathcal{A} \rightarrow$ **w** **r**

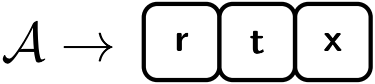
Breadth First Search



$\mathcal{A} \rightarrow$ **r** **t**

Breadth First Search





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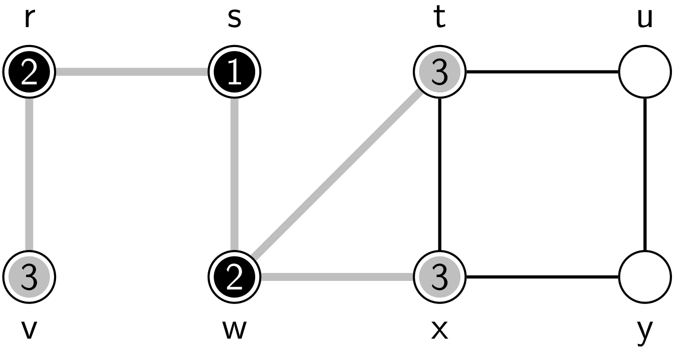

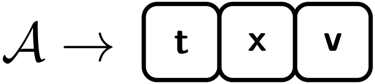

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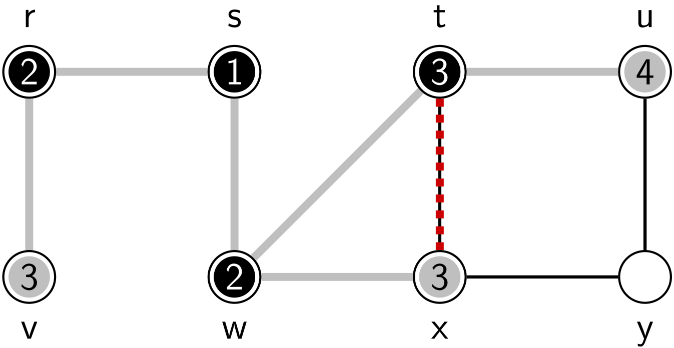
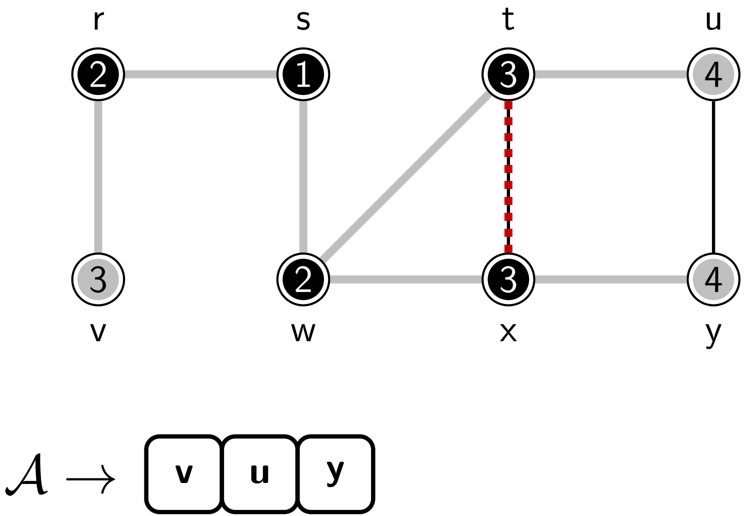


$\mathcal{A} \rightarrow$ **x** | **v** | **u**

Breadth First Search



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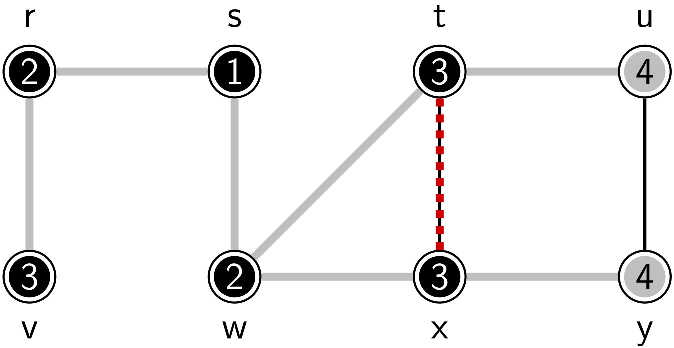
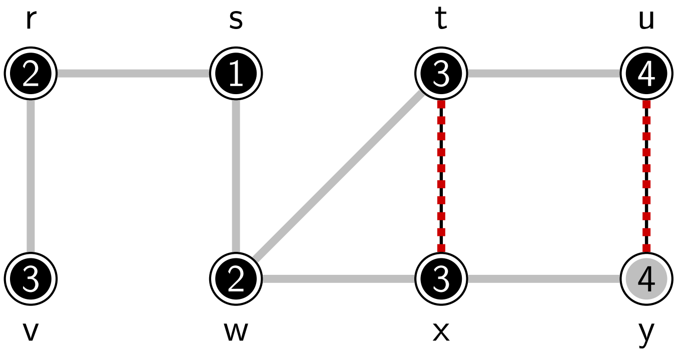


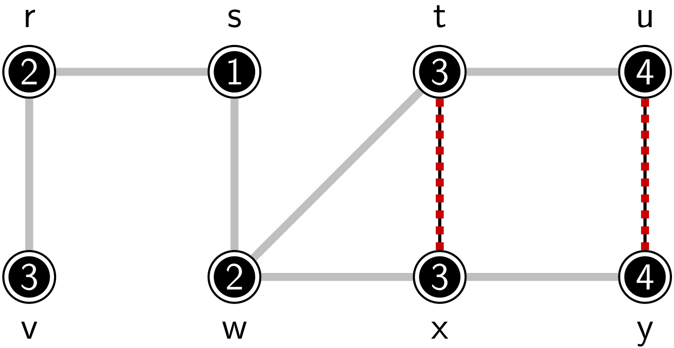
$\mathcal{A} \rightarrow$ **u** **y**

Breadth First Search



$\mathcal{A} \rightarrow$ **y**

Breadth First Search



$\mathcal{A} \rightarrow$

Properties of BFS

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- They obviously form a tree
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- Grey edges mark first discoveries of neighbor nodes
- They obviously form a tree
- Do you have an interpretation for $d(v)$?
- In fact, $d(v) - 1$ is the minimal distance from the startnode
- The (grey) tree edges are minimal length paths

DFS: Recursive Procedure

DFS(g):

for all $v \in \mathcal{V}$ **do**
  $d(v) = 0$
$time = 1$ // the time when a vertex is touched
for all $v \in \mathcal{V}$ **do**
  if $d(v) == 0$ **then**
    DFS-Visit(v)

DFS-Visit(v):

$d(v) = time; time = time + 1$ // v gets grey
for all $e = (v, u)$ **do**
  if $d(u) == 0$ **then**
    DFS-Visit(u) // is u white? Then visit it
$f(v) = time; time = time + 1$ // v gets black

Edge Classification using DFS

We store two timestamps for each vertex v

- the discovery time $d(v)$, when v changes from white to grey
- the finishing time $f(v)$, when v changes from grey to black

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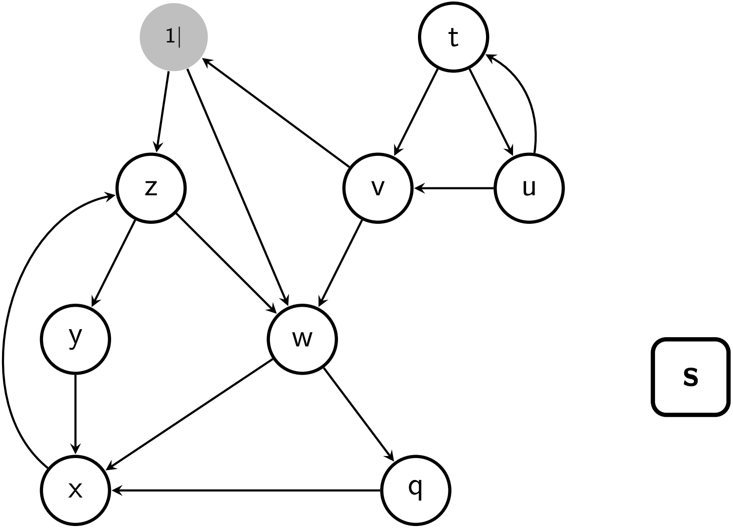
The edges of a directed graph can be classified into four categories, depending on the role they play in a run of depth first search.

- tree edges: the edges used in the recursion (ending on a white vertex)
- backward edges: edges ending in a grey vertex (including self loops)
- forward edges: edges (n, m) ending in a black vertex, and $d[n] < d[m]$
- cross edges: edges (n, m) ending in a black vertex, and $d[m] < d[n]$

DFS example



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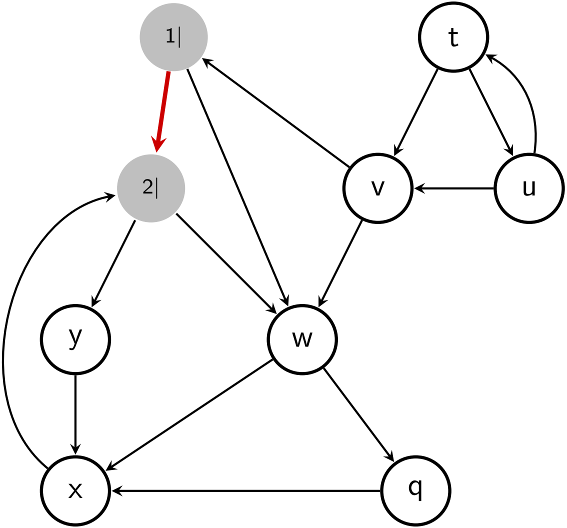

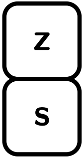

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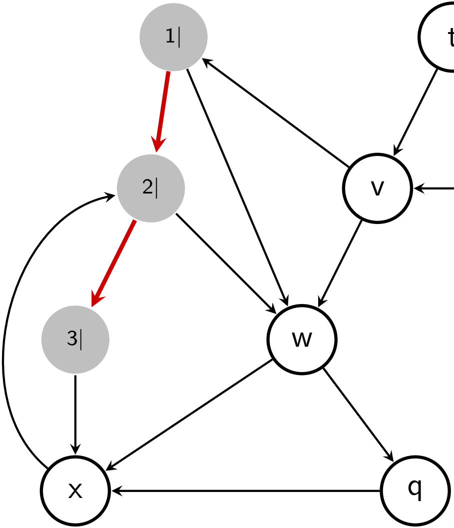

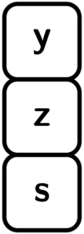

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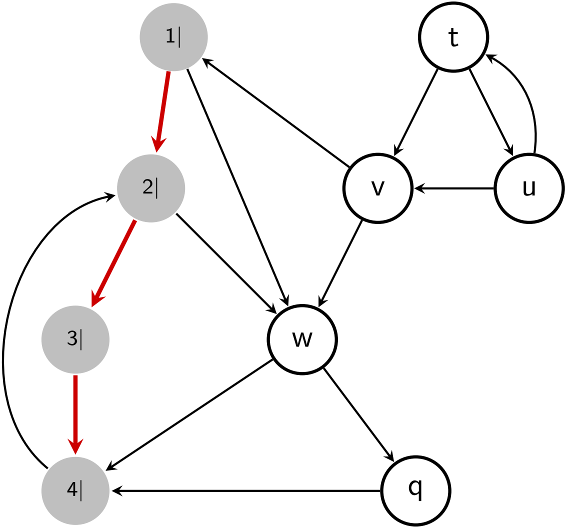

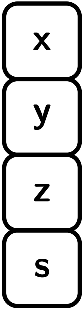

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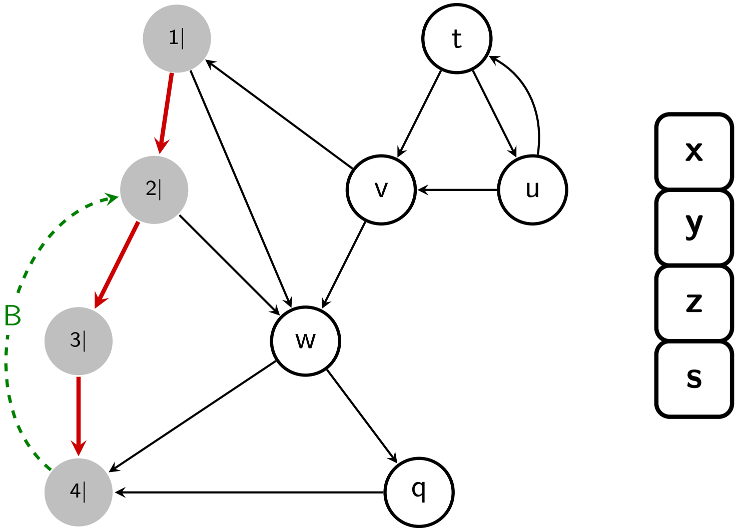

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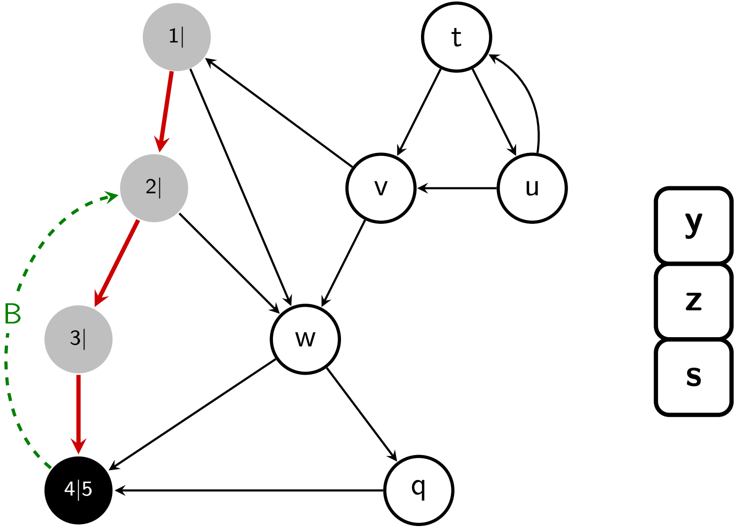

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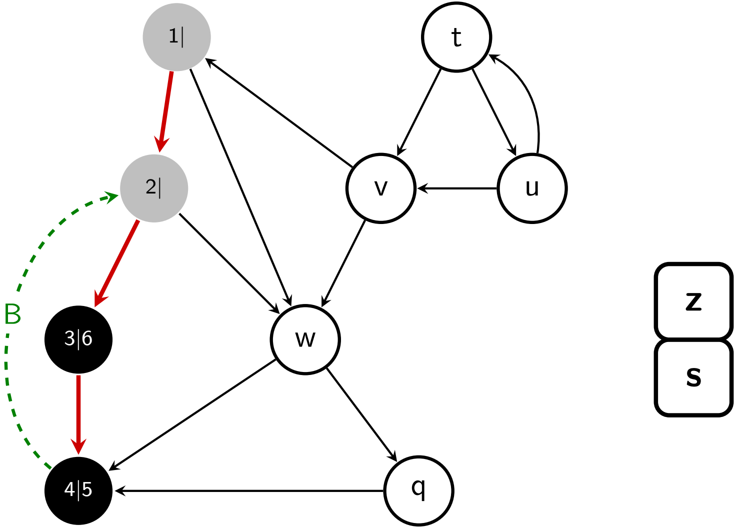

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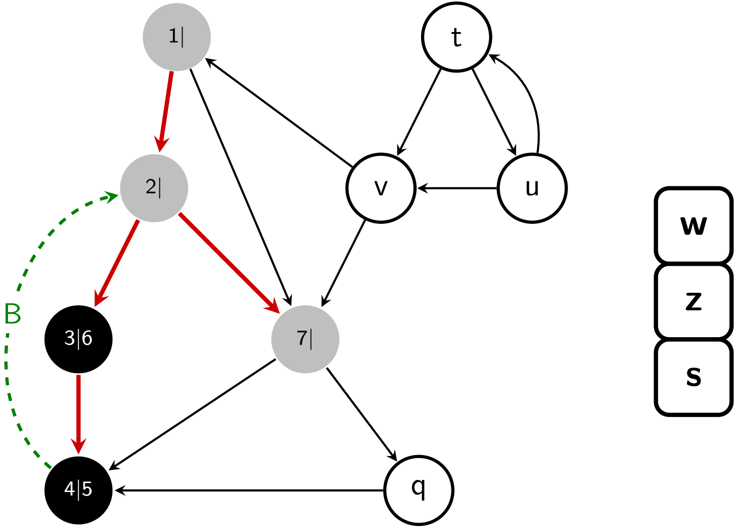

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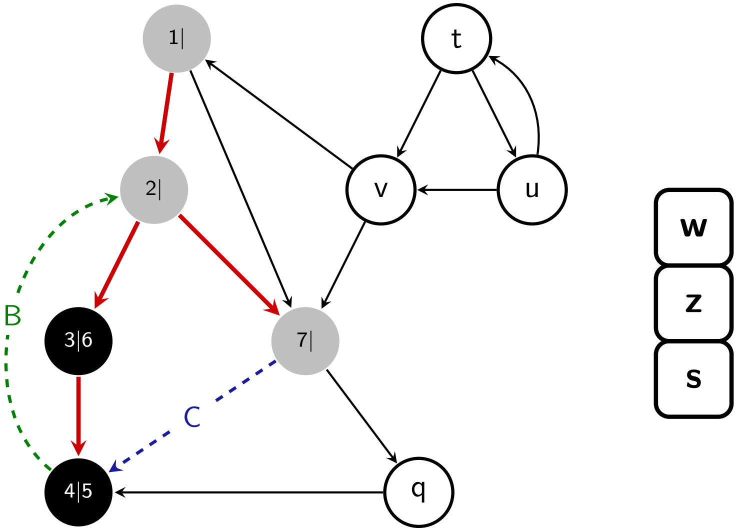

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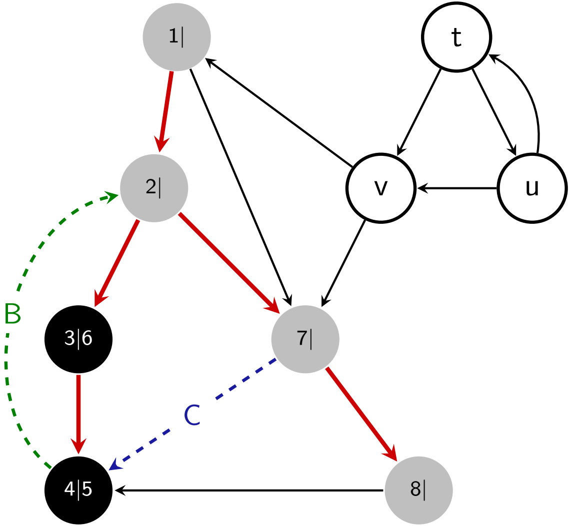

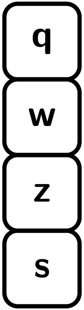

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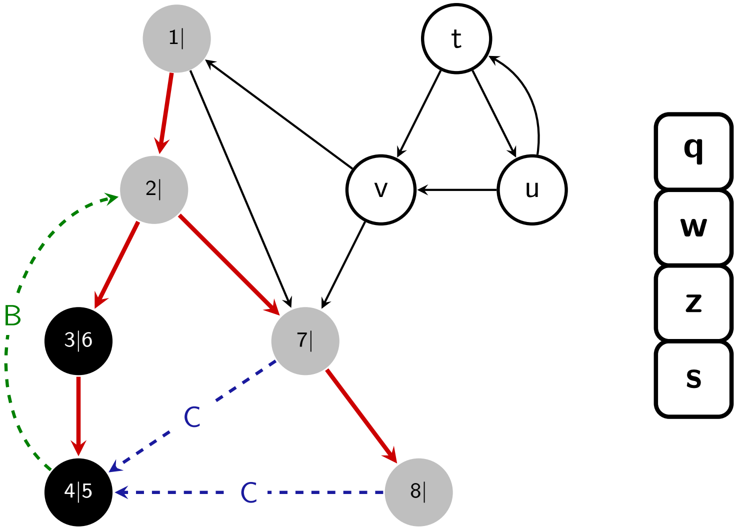

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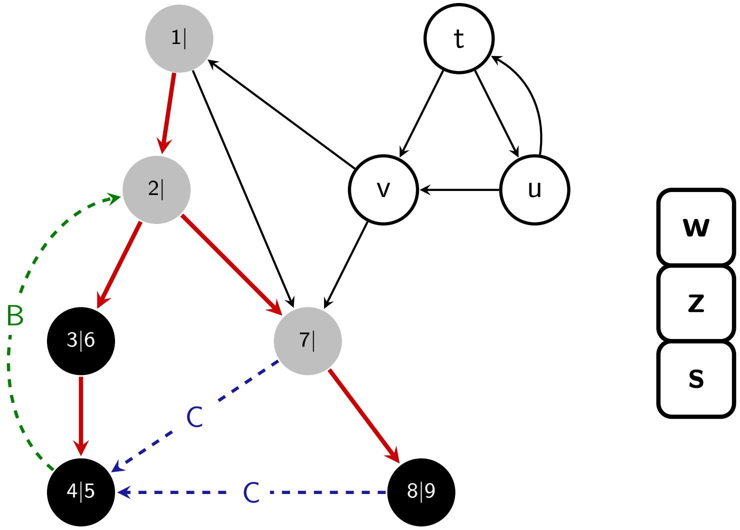

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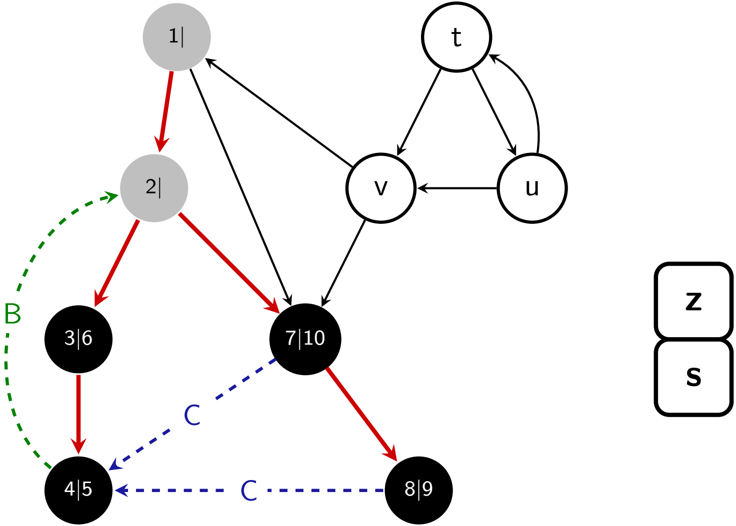

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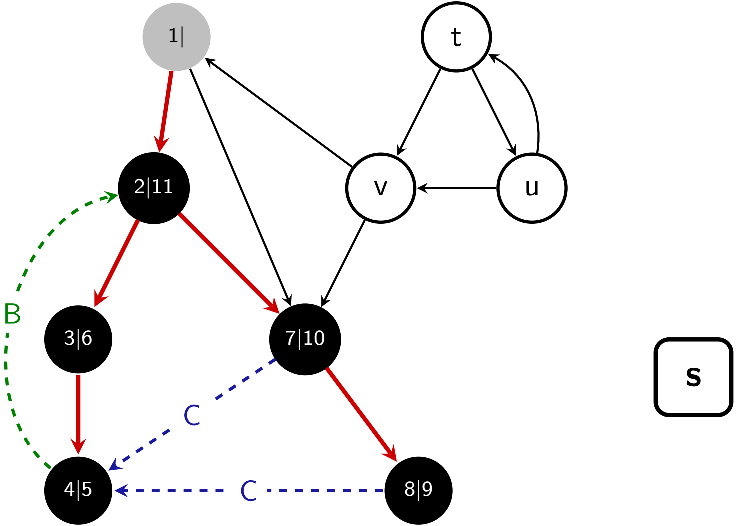

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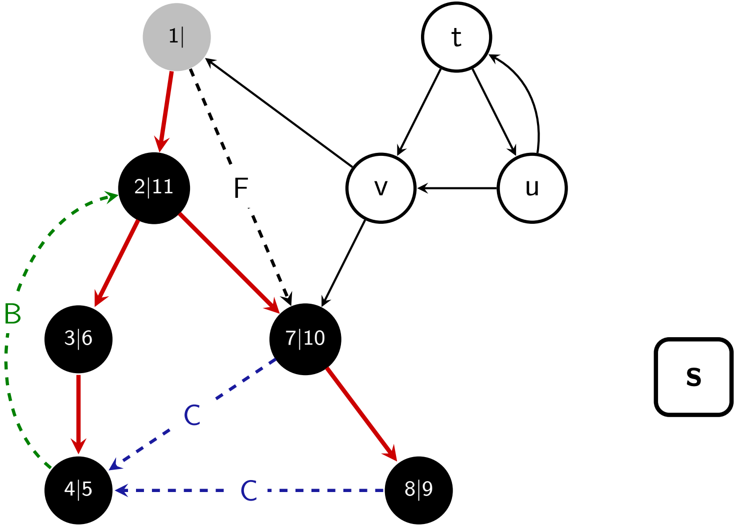

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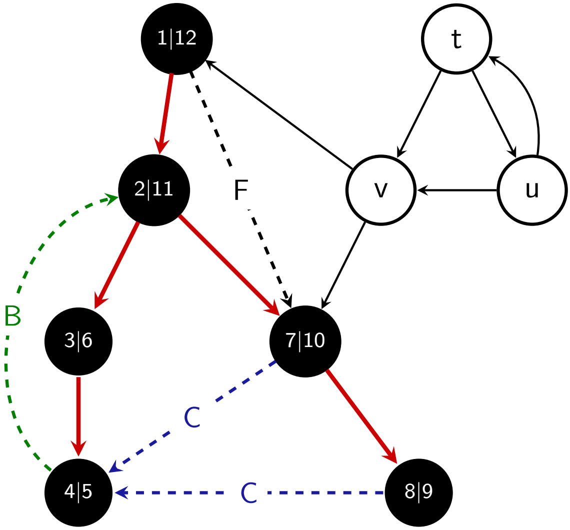

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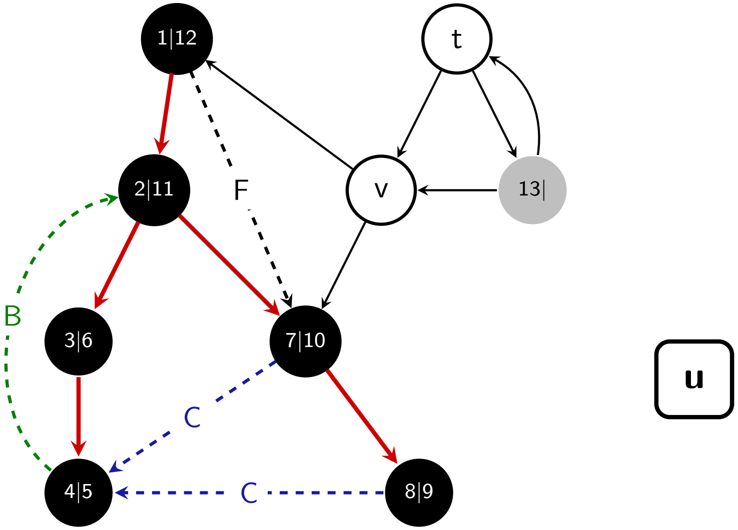

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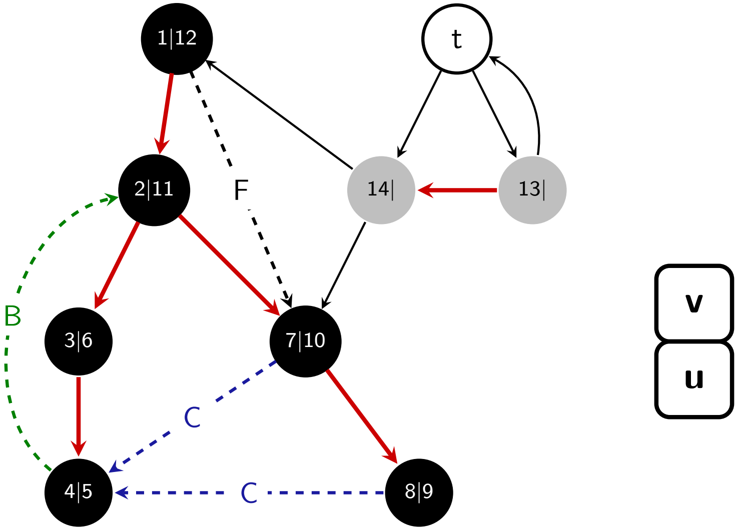

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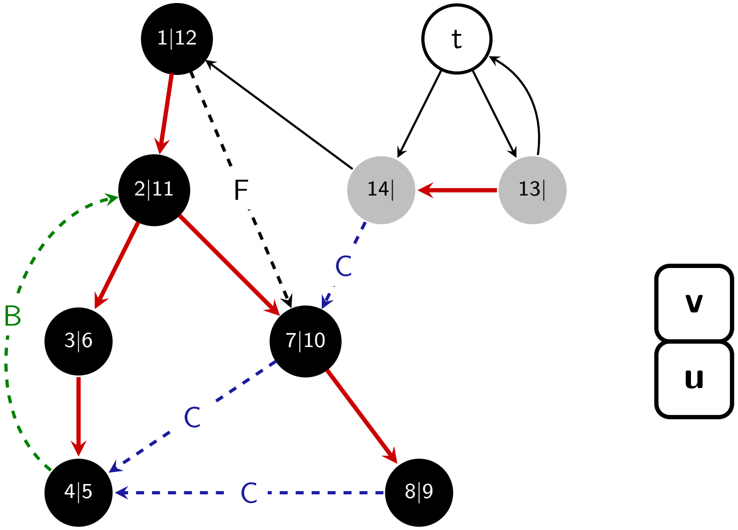

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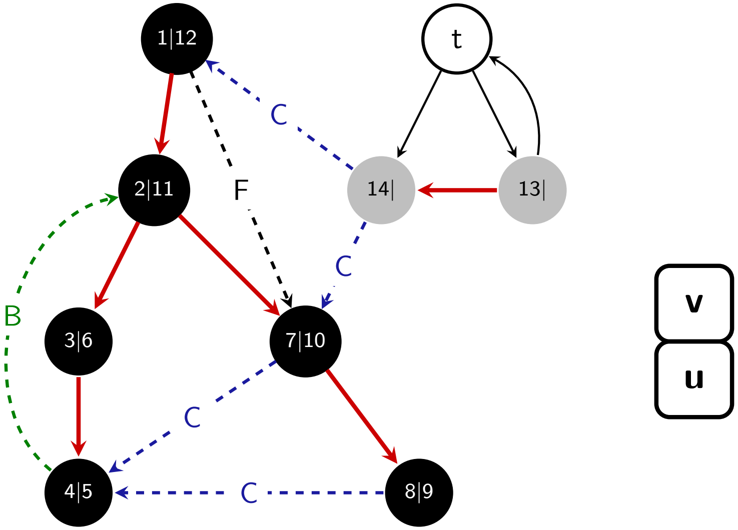

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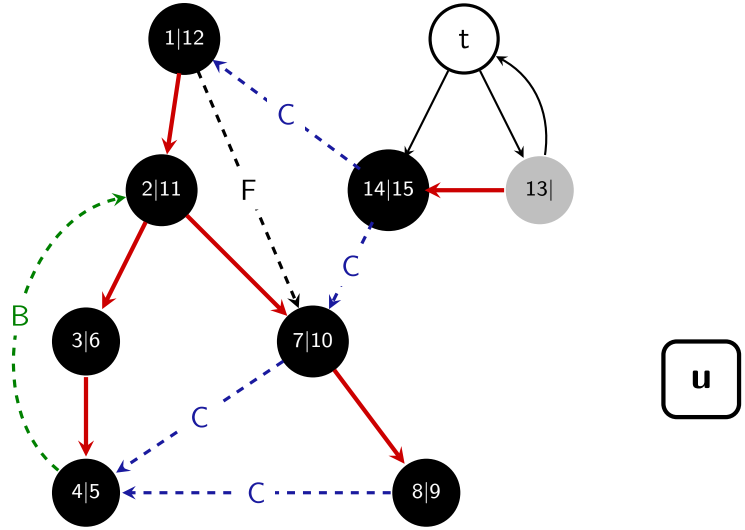

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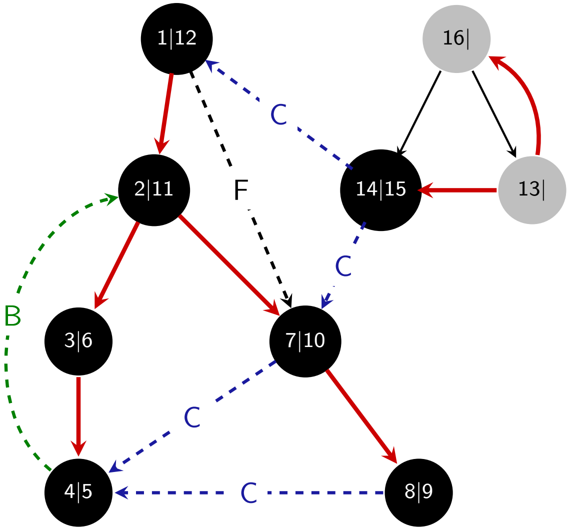

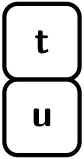

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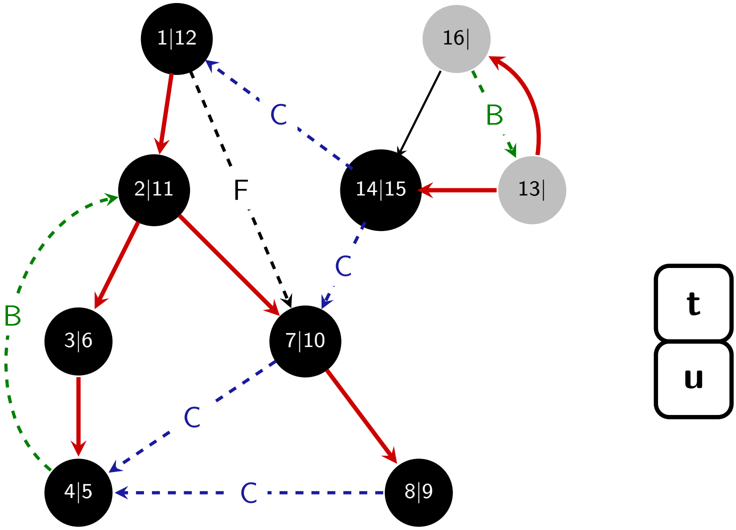

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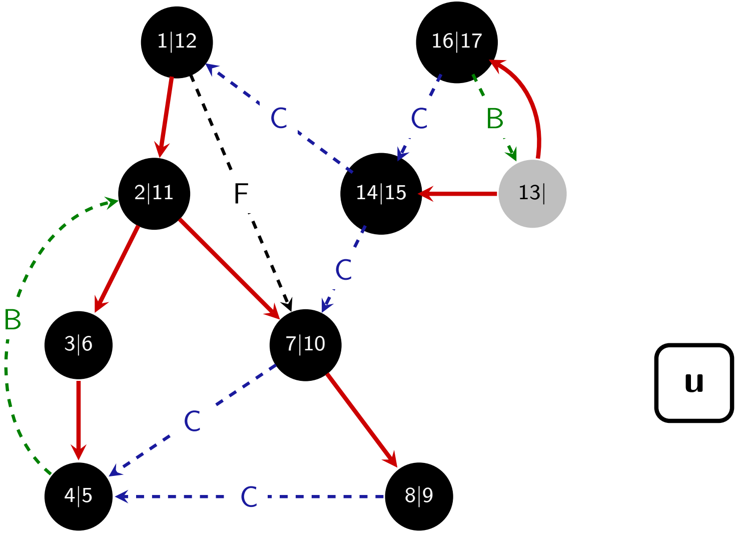

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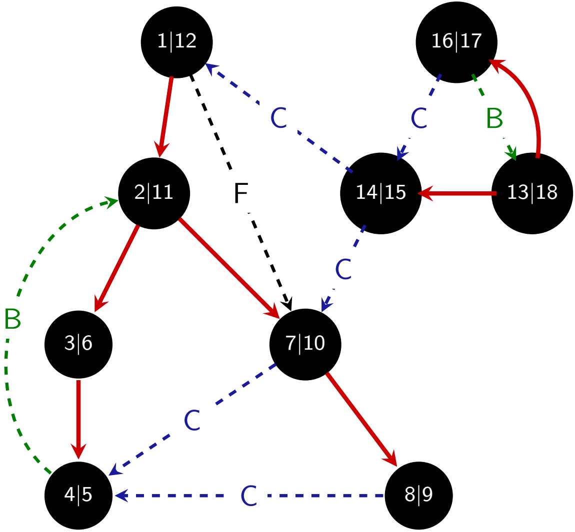
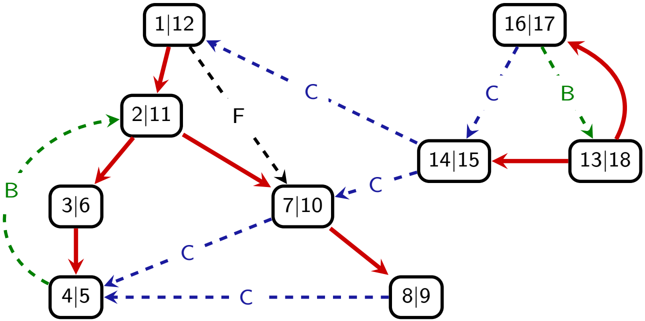


Edge Classification Example



During DFS, edge (s, t) is

Tree:	$f(t) = 0 \wedge d(s) < d(t)$	Back:	$f(t) = 0 \wedge d(s) > d(t)$
Cross:	$f(t) \neq 0 \wedge d(t) < d(s)$	Forward:	$f(t) \neq 0 \wedge d(t) > d(s)$

After DFS: $f(t) \neq 0 \Rightarrow f(t) < f(s)$

Note: a directed graph is acyclic if there are no back edges.

DFS/BFS Visitors

- Recap: Visitor Pattern, a Behavioural Pattern
- Purpose: Add functionality to a class without changing it
- Implementation:
  - Methods of class A get a visitor object as argument
  - The visitor's interface methods are called at specific points of the computation and have an A object as argument (at least)
  - This allows different additional computations or side effects with one class method of class A
  - The functionality is parameterized by the different visitor objects and classes, so to speak
- Especially useful with traversal methods of complex structures (like DFS or BFS)
- The graph implements the traversal, the visitor the special functionality

DFS/BFS Visitors II

- Methods common for DFS/BFS visitor interface
  - `startNode(v,g)` : v is white in the outer loop
  - `discoverNode(v,g)` : v changes from white to gray
  - `finishNode(v,g)` : v changes from gray to black
  - `nonTreeEdge(e,g)` : visit edge with gray or black target node
- Methods specific to BFS visitor
  - `treeEdge(e,g)` : visit edge with white target node
- Methods specific to DFS visitor:
  - `treeEdgeBefore(e,g)` : before recursion into `target`(e)
  - `treeEdgeAfter(e,g)` : after recursion into `target`(e)

Topological Order

- Order the vertices such that if (n, m) is an edge, n comes before m
- Only exists for acyclic graphs
- Algorithm: sort vertices according to decreasing finishing times of DFS
- This can be easily implemented by a DFS visitor
- As each vertex is finished, add it to the front of a linked list
- The visitor contains this list as member variable
- After all vertices have been visited by DFS, the visitor holds the result
- Application example: A constraint graph that locally specifies which action must precede another

Topological Order Example