Java II
Graphs and Search

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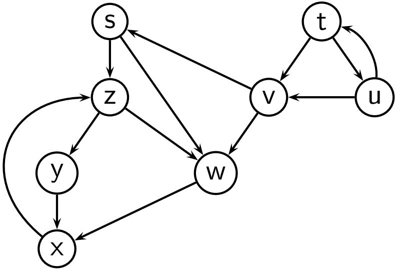
Graphs: Definition

- Graph $\mathcal{G}$: A set of vertices (nodes) $\mathcal{V}$ and a set of edges $\mathcal{E}$, which is a relation on vertices, that is: $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$
- Example:
  - Vertices: students at the university
  - $(u, v) \in \mathcal{E} \Leftrightarrow$ student u knows student v
- Graphical representation:
  - vertices: blobs
  - edges: arrows (arcs) between the blobs
- If $\mathcal{E}$ is symmetric, i.e., if $(u, v) \in \mathcal{E} \Leftrightarrow (v, u) \in \mathcal{E}$ the graph is called *undirected* (plain arcs, not arrows)
- Example: $\mathcal{E}$ is the set of students that are akin

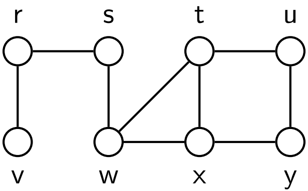
Graphs: Definitions II

- Vertex u is *reachable* from vertex v ($u \rightarrow v$) iff there is a sequence of edges $(u, w_1), (w_1, w_2), \ldots, (w_n, v)$ in $\mathcal{E}$
- A graph is *cyclic* (contains a cycle) if there is $u \in \mathcal{V}$ s.th. $u \rightarrow u$ over a nontrivial sequence of edges in $\mathcal{E}$, including a self loop

directed graph



undirected graph



Implementation Basics

- Represent the vertices as numbers from zero to $|\mathcal{V}| - 1$
- Matrix representation: represent $\mathcal{E}$ as a quadratic `boolean` matrix A of size $|\mathcal{V}|$; $A[i, j]$ is `true` iff $(i, j) \in \mathcal{E}$

+ Good for dense graphs, where $|\mathcal{E}| \approx |\mathcal{V}|^2$: only one bit per edge
+ Fast: are two vertices directly connected?

– Initialization is quadratic in $|\mathcal{V}|$
– Visiting all outgoing edges of a vertex takes $|\mathcal{V}|$ steps, no matter how many there really are
– Additional information attached to the edges (e.g., weights) has to be stored separately

Adjacency List Representation

- For every vertex, store a list of outgoing edges, i.e., the vertex number that is reached
- Graph is represented by an array of list heads
- In Java: `ArrayList` of `Lists`.
- \+ Compact representation for most graphs, except if they are very dense
- \+ Allows more efficient implementations of many graph algorithms
- \+ Additional edge information can be stored in the elements of the edge lists directly

Search in Graphs

- Task: visit all reachable vertices, starting at vertex s
- Iteratively use all the outgoing edges of s, and all the nodes that can be reached through these egdes
- Make sure that no node gets explored twice
- Basic idea: maintain two sets
  - $\mathcal{U}$ the *visited* nodes
  - $\mathcal{A}$ the *active* nodes, i.e., still unexplored outedges
- In textbooks, vertices are often assigned colors during the search:
  - White: not in $\mathcal{U}$ and not in $\mathcal{A}$
  - Grey: in $\mathcal{U}$ and in $\mathcal{A}$ (under consideration)
  - Black: in $\mathcal{U}$, but not in $\mathcal{A}$ anymore (finished)

Generic Search Algorithm

- Initialization: both sets contain only the start vertex s

proc *Search*(s) $\equiv$
  $\mathcal{U} = \mathcal{A} = \{s\}$ // s gets grey
  while $\mathcal{A} \neq \emptyset$ **do**
    for some node $n \in \mathcal{A}$
      if there is an unused edge $e = (n, m)$ leaving n
        if $m \notin \mathcal{U}$ **then** // m gets grey
          $\mathcal{U} = \mathcal{U} \cup \{m\}$; $\mathcal{A} = \mathcal{A} \cup \{m\}$
      else
        $\mathcal{A} = \mathcal{A} - \{n\}$ // n gets black

- Questions:
  - How to implement sets $\mathcal{U}$ and $\mathcal{A}$?
  - Does the result depend on the implementation?

Implementation of $\mathcal{U}$

- What is the best data structure for $\mathcal{U}$?
- What are the operations on $\mathcal{U}$?

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  2. Is node n contained in the set?
- $\mathcal{U}$ should be implemented as a bit vector over the nodes
- Two alternatives:
  - `boolean` member variable of the node data structure
  - A so-called *property vector* (or *property map*) attached to the vertices

Property Vectors

Advantages and drawbacks of property vectors

- More flexible:
  - Create all and only those you need for an algorithm
  - In a graph framework, one can not put all the data into the vertices
  - May contain any type, small or bigger datastructures
  - Only use memory when they are needed
- Require an efficient indexing between vertices and values: maintain a numeric index in the vertices
- Member variables are always faster

Property vectors can also be used for graph edges

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The choice of the data structure for $\mathcal{A}$ and the decisions about n and e determine the order in which vertices are visited

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  - Add a vertex
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  - Test if set is empty

Implementation of $\mathcal{A}$

The choice of the data structure for $\mathcal{A}$ and the decisions about *n* and *e* determine the order in which vertices are visited

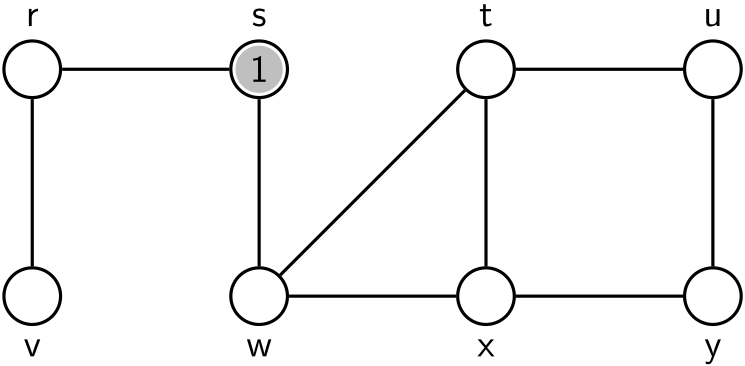
- Operations on set $\mathcal{A}$:
  - Add a vertex
  - Get and remove some vertex (nondeterministic)
  - Test if set is empty
- Implement $\mathcal{A}$ as a *queue* and keep *n* until it gets black: Breadth First Search (BFS)
- Implement $\mathcal{A}$ as a *stack* and always take its top element: Depth First Search (DFS)
- DFS is often implemented as a recursive function, the function call stack takes the role of $\mathcal{A}$

BFS Implementation

```
proc BFS() ≡
  foreach v ∈ 𝒱 do d(v) = 0
  time = 1          // the time when a vertex is touched
  foreach v ∈ 𝒱 with d(v) == 0 do        // v is the start node
    d(v) = time; 𝒜.push_back(v)          // v gets grey
    while ¬ 𝒜.empty() do
        n = 𝒜.pop_front()
        time = d(n) + 1
        foreach e = (n, m) do
          if d(m) == 0 then          // m ∉ 𝒰 ?
            d(m) = time; 𝒜.push_back(m)          // m gets grey
        // n gets black
```

- Finally, all vertices of $\mathcal{G}$ have been visited
- The $d(v)$ is abused to serve as the $\mathcal{U}$ bitvector

Breadth First Search



$\mathcal{A} \rightarrow$

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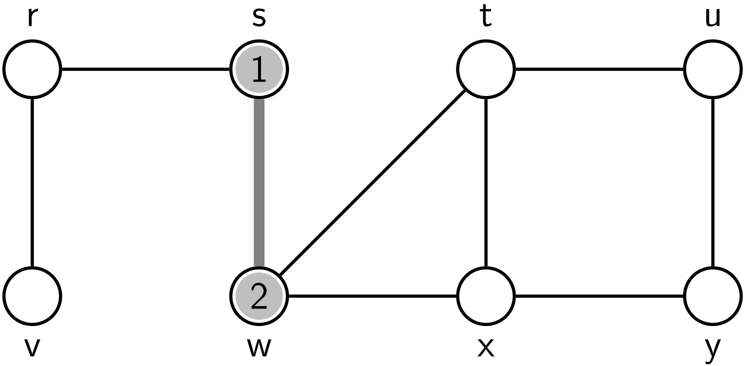

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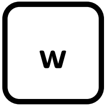

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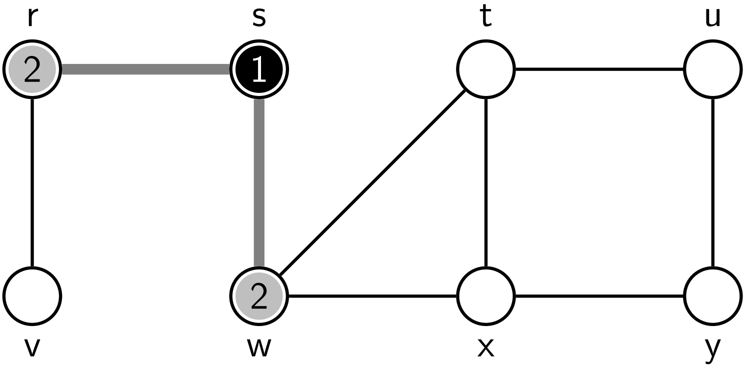

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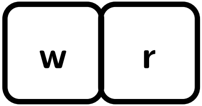

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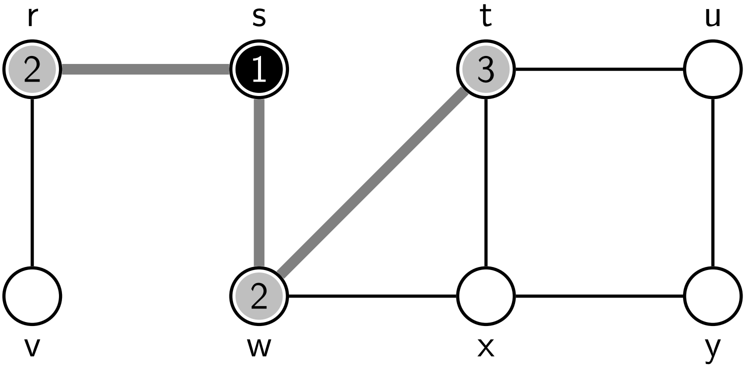

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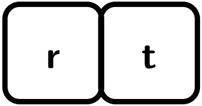

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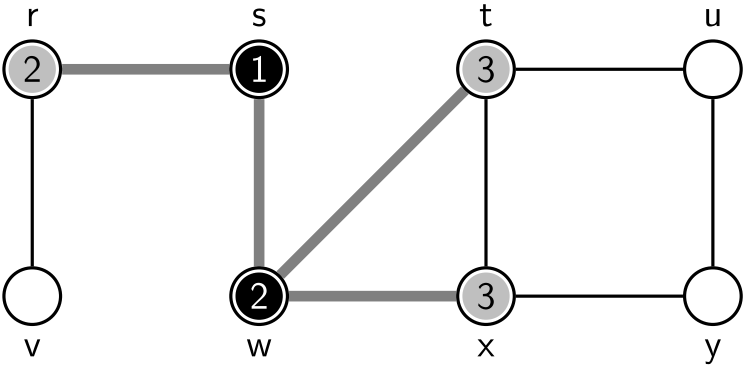

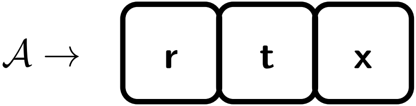

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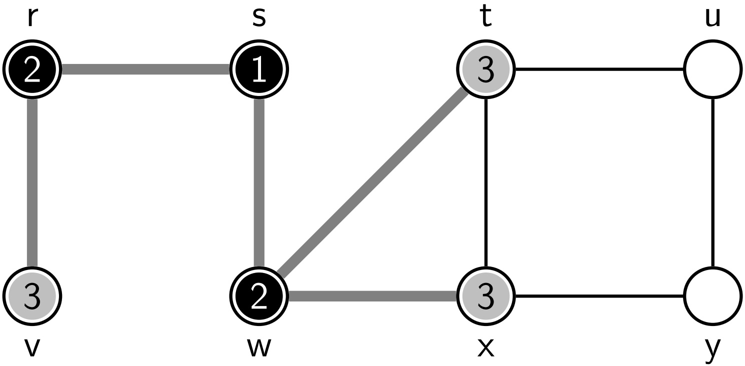
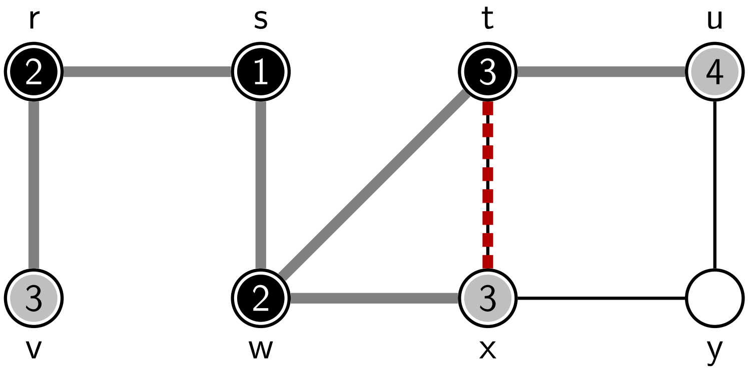


$\mathcal{A} \rightarrow$ | t | x | v |

Breadth First Search



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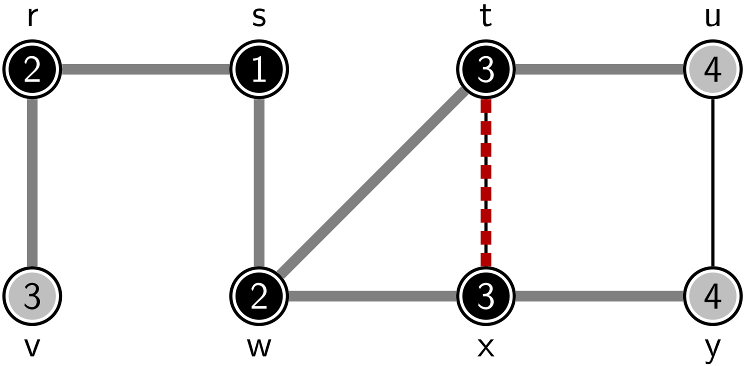

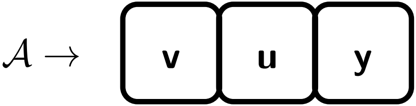

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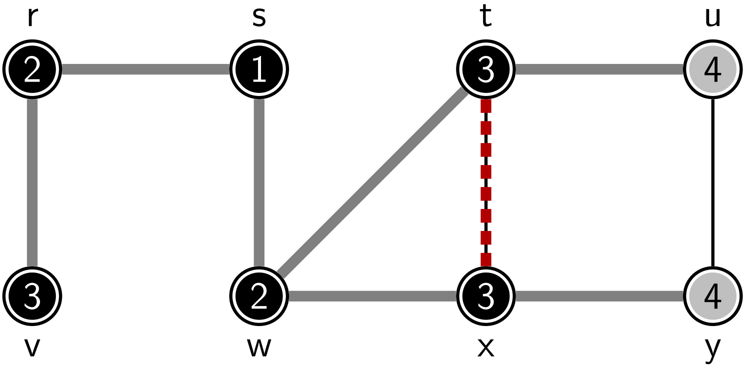

$\mathcal{A} \rightarrow$ **u** **y**

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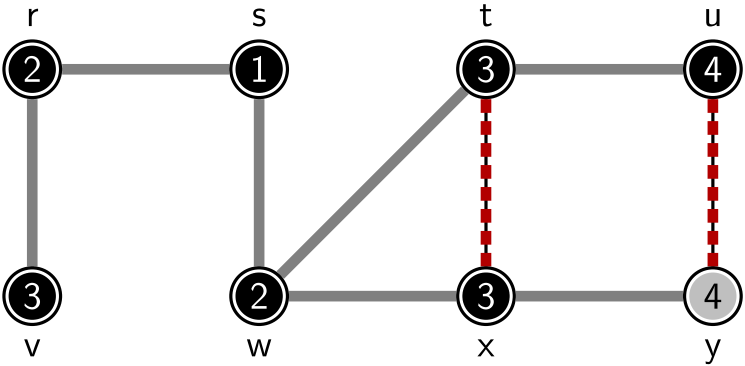

$\mathcal{A} \rightarrow$ **y**

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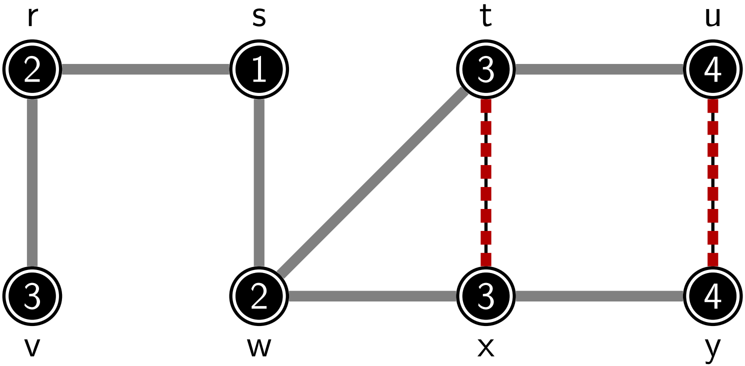

$\mathcal{A} \rightarrow$

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- Grey edges mark first discoveries of neighbor nodes
- They obviously form a tree
- Do you have an interpretation for $d(v)$?
- In fact, $d(v) - 1$ is the *minimal distance* from the startnode
- The (grey) tree edges are minimal length paths

DFS: Recursive Procedure

$\underline{\textbf{proc}}\ \mathit{DFS}(G)\ \equiv$
$\quad \underline{\textbf{foreach}}\ v \in \mathcal{V}\ \underline{\textbf{do}}\ d(v) = 0$
$\quad \mathit{time} = 1$ // the time when a vertex is touched
$\quad \underline{\textbf{foreach}}\ v \in \mathcal{V}\ \underline{\textbf{do}}\ \underline{\textbf{if}}\ d(v) == 0\ \underline{\textbf{then}}\ \mathit{DFS\text{-}Visit}(v)$

$\underline{\textbf{proc}}\ \mathit{DFS\text{-}Visit}(v)\ \equiv$
$\quad d(v) = \mathit{time};\ \mathit{time} = \mathit{time} + 1$ // v gets grey
$\quad \underline{\textbf{foreach}}\ e = (v, u)\ \underline{\textbf{do}}$
$\quad\quad \underline{\textbf{if}}\ d(u) == 0\ \underline{\textbf{then}}\ \mathit{DFS\text{-}Visit}(u)$ // is u white? Then visit it.
$\quad f(v) = \mathit{time};\ \mathit{time} = \mathit{time} + 1$ // v gets black

We store two timestamps for each vertex v

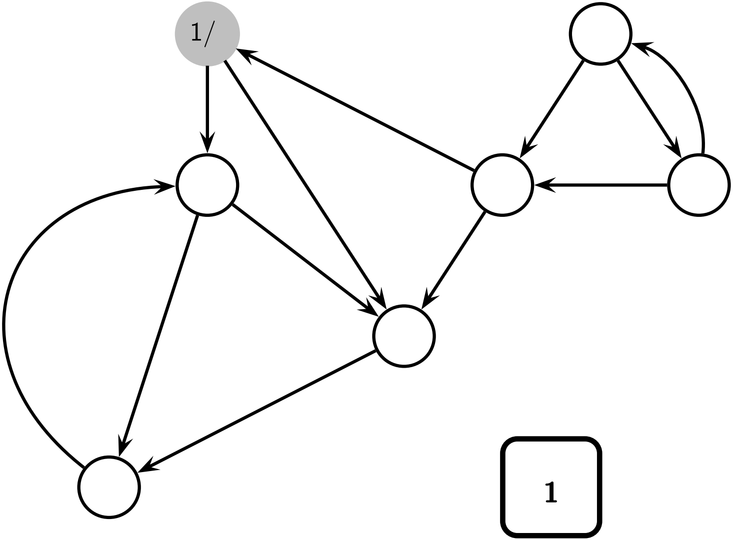
- the *discovery time* $d(v)$, when v changes from white to grey
- the *finishing time* $f(v)$, when v changes from grey to black

Edge Classification using DFS

The edges of a directed graph can be classified into four categories, depending on the role they play in a run of depth first search.

- tree edges: the edges used in the recursion (ending on a white vertex)
- backward edges: edges ending in a grey vertex (including self loops)
- forward edges: edges (n, m) ending in a black vertex, and $d[n] < d[m]$
- cross edges: edges (n, m) ending in a black vertex, and $d[m] < d[n]$

DFS example



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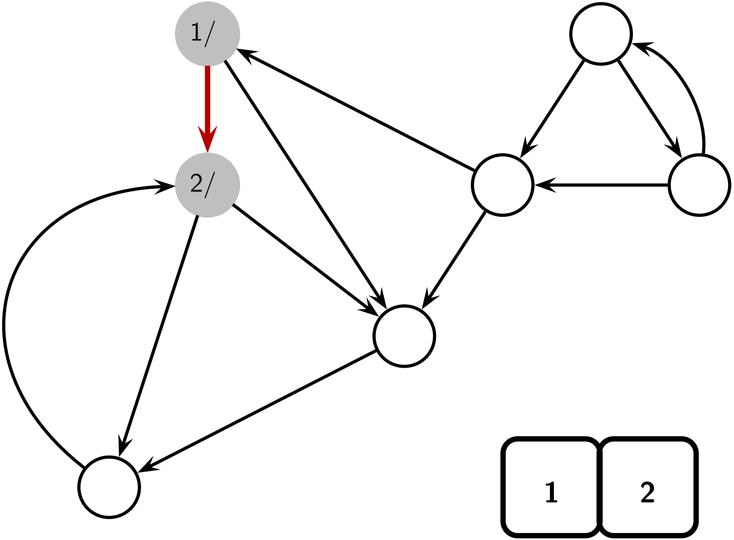

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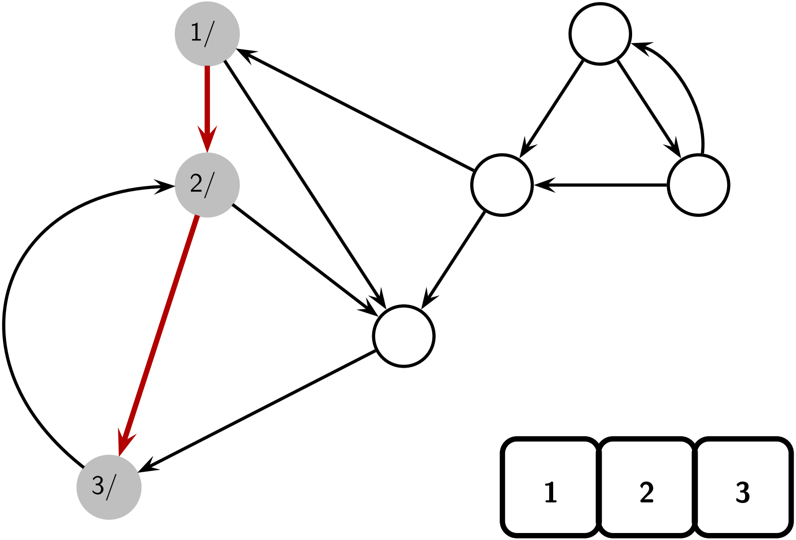

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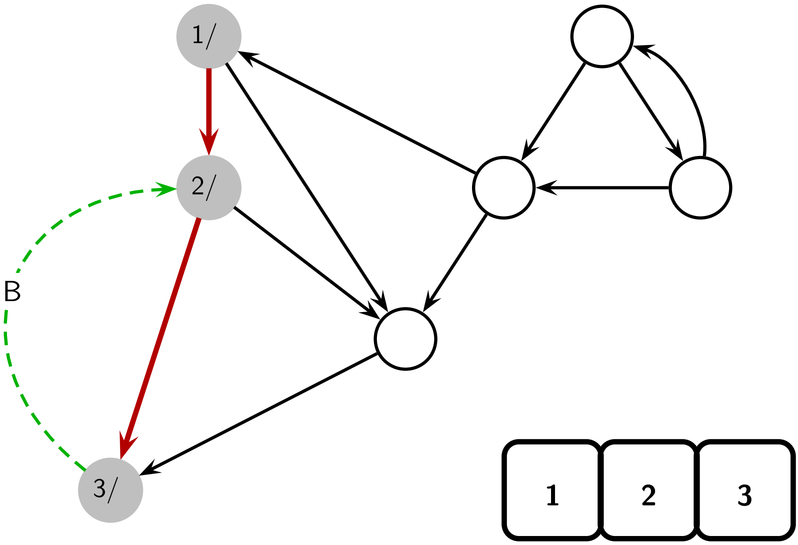

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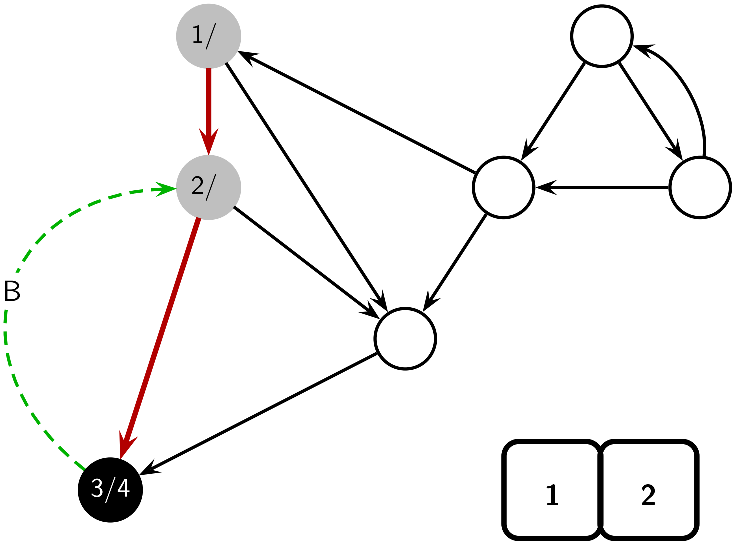

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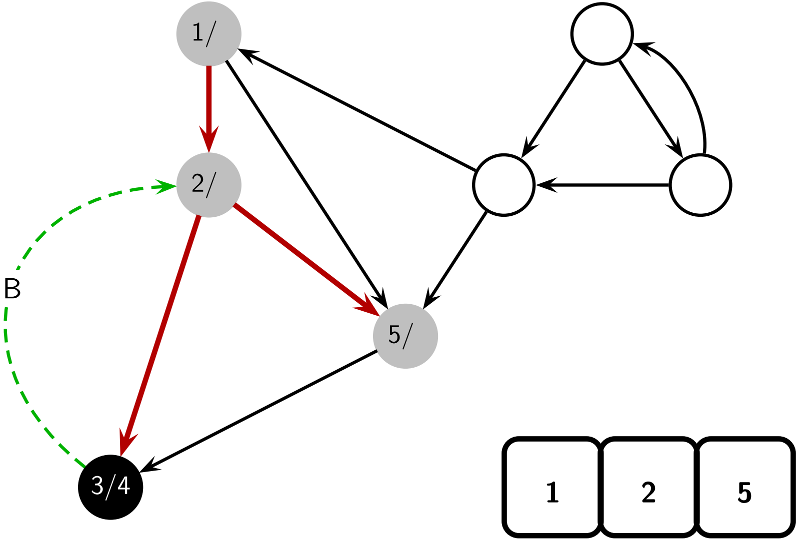

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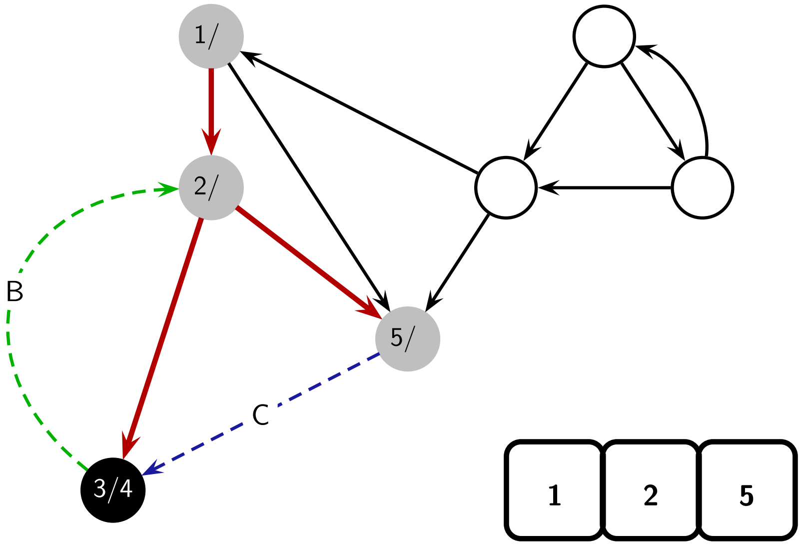

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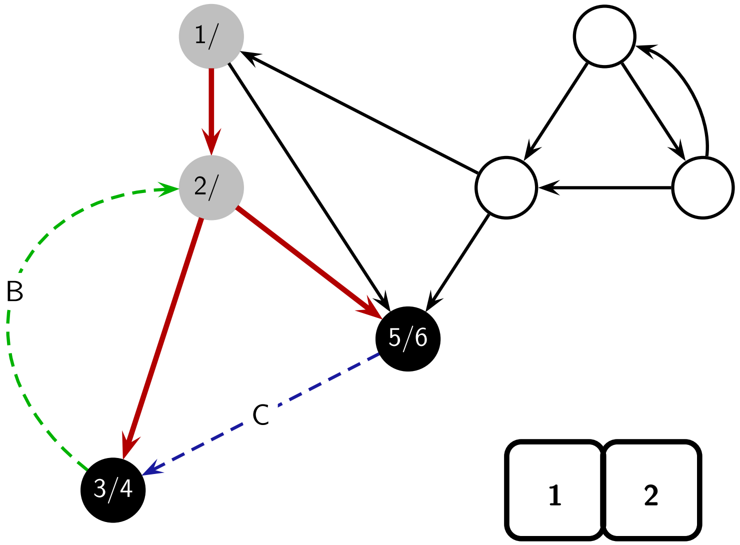

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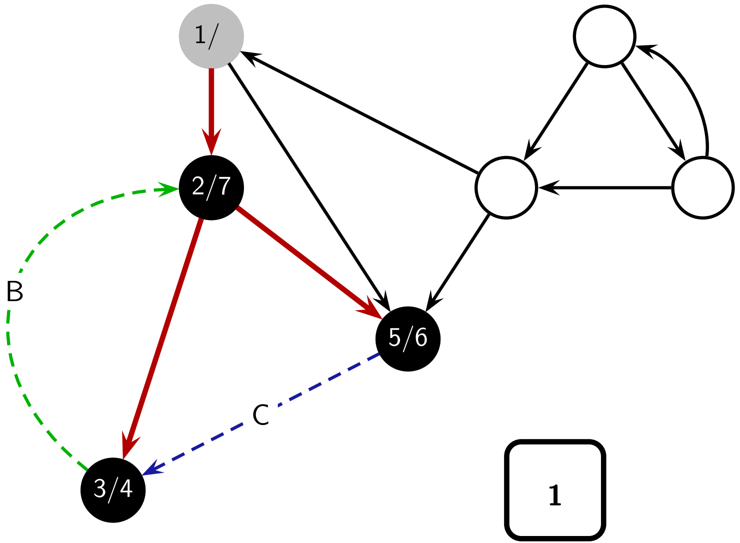

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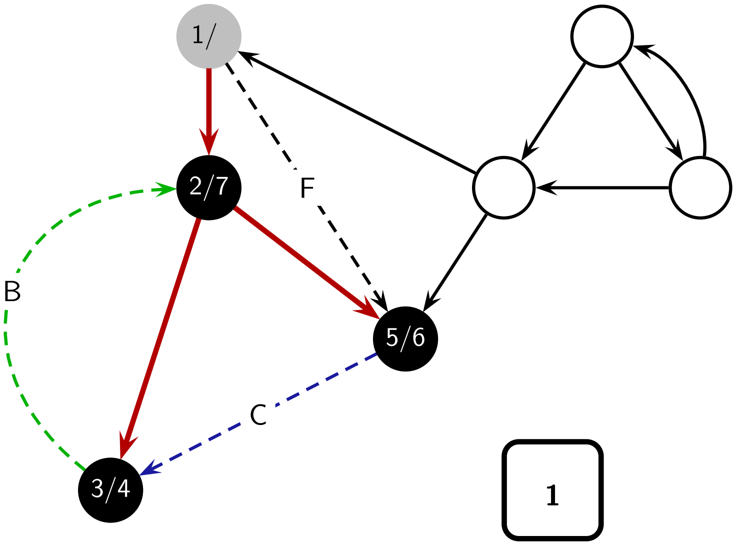

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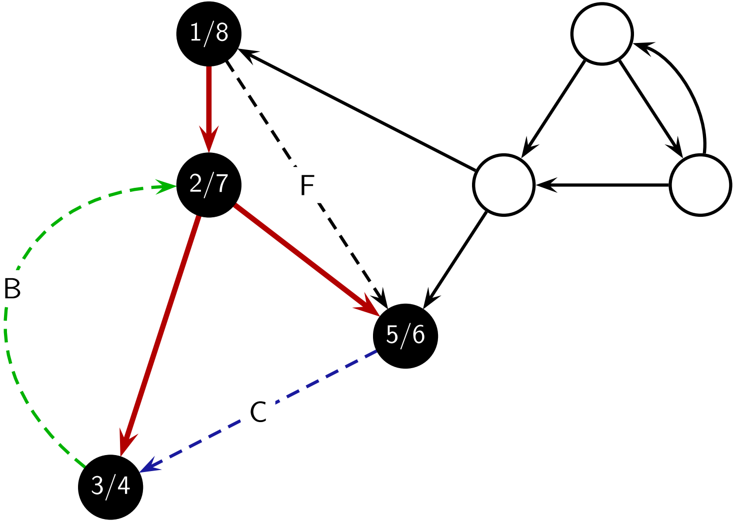

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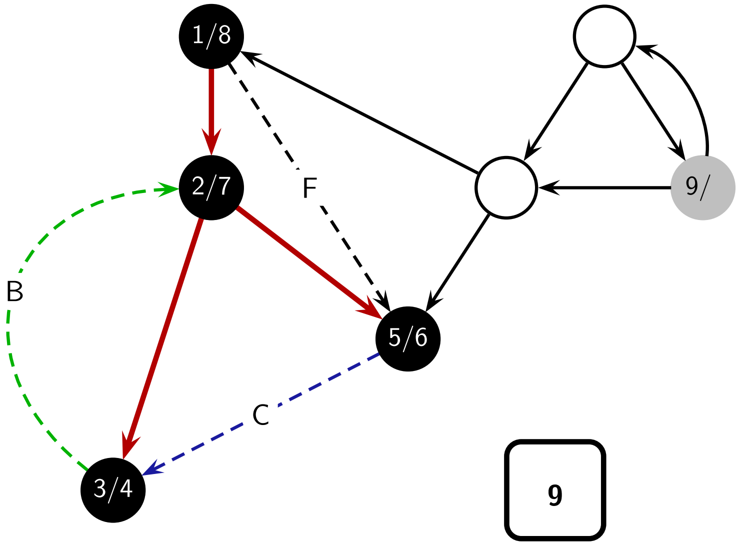

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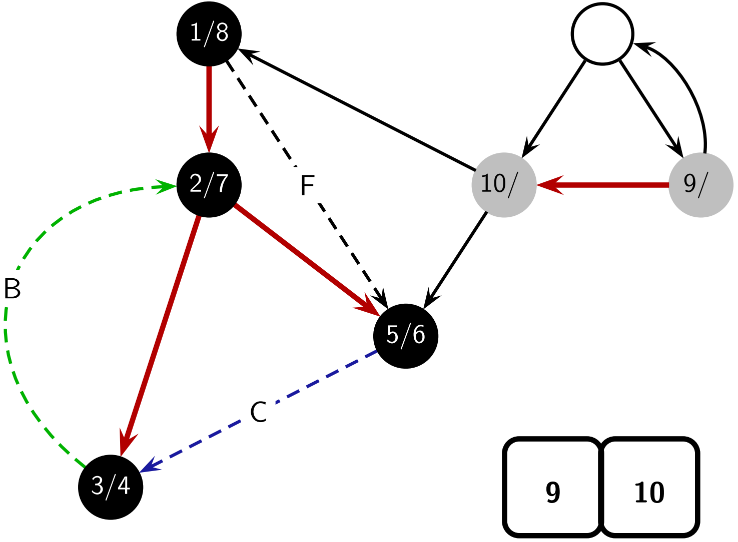

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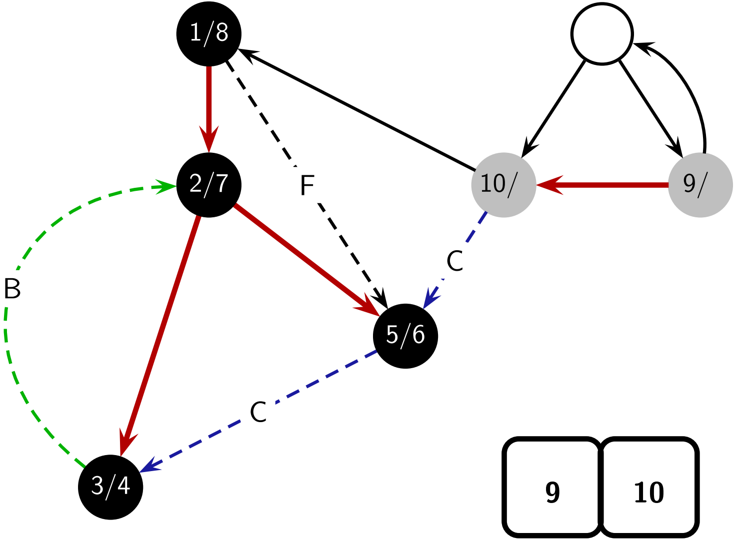

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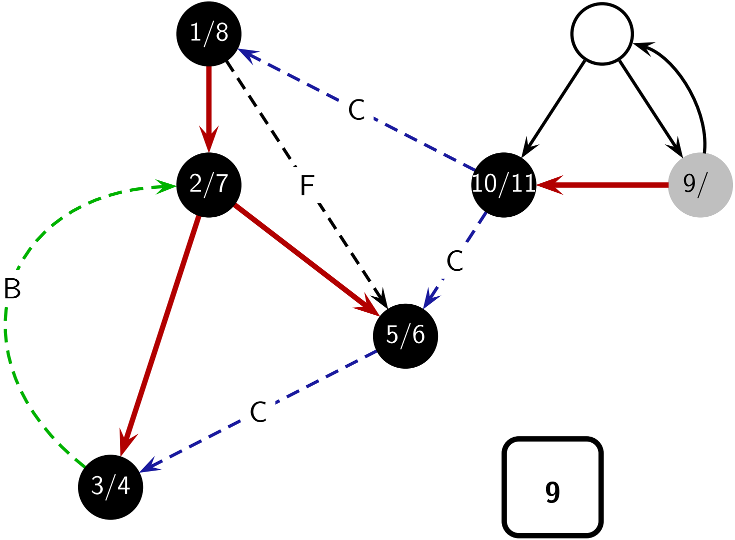

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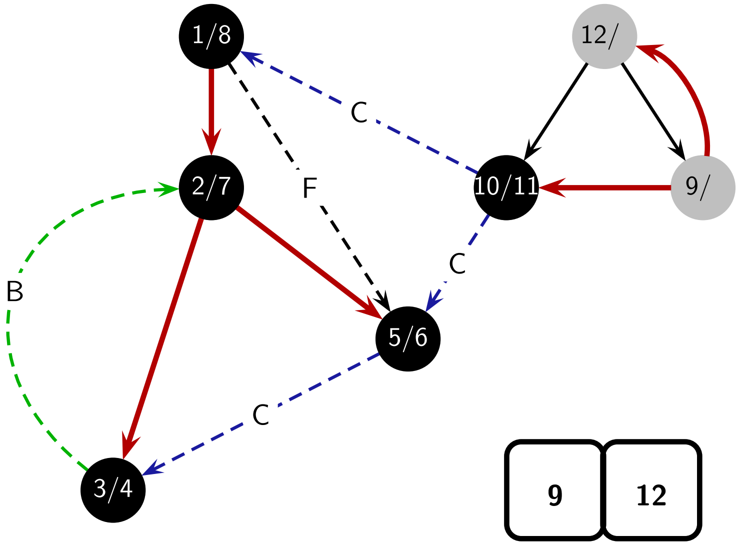

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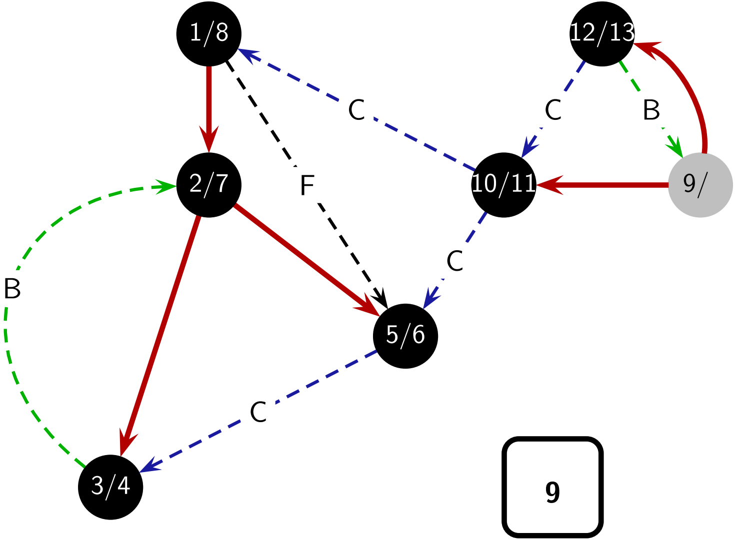

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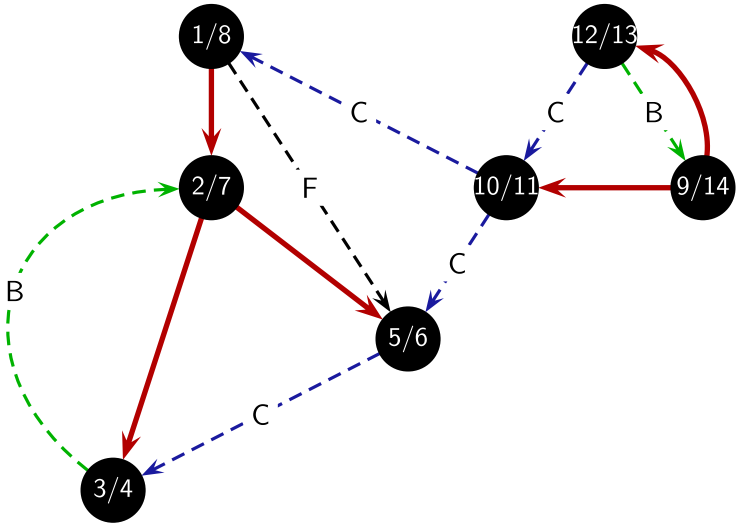
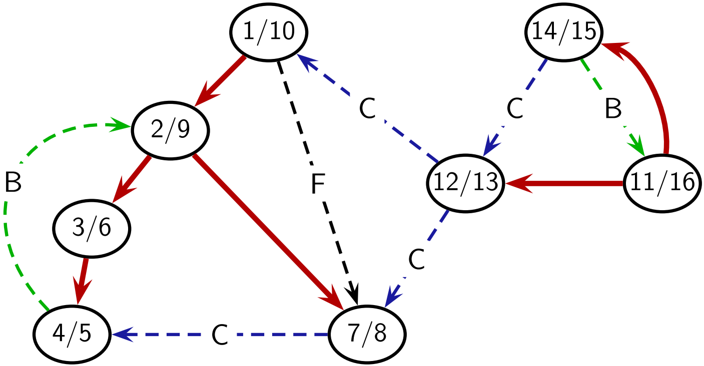


Edge Classification Example



During DFS, edge (s, t) is

Tree: $f(t) = 0 \wedge d(s) < d(t)$ Back: $f(t) = 0 \wedge d(s) > d(t)$

Cross: $f(t) \neq 0 \wedge d(t) < d(s)$ Forward: $f(t) \neq 0 \wedge d(t) > d(s)$

After DFS: $f(t) = 0 \Rightarrow f(t) > f(s)$ and $f(t) \neq 0 \Rightarrow f(t) < f(s)$

Note: a directed graph is acyclic if there are no back edges.

DFS/BFS Visitors

- Recap: Visitor Pattern, a Behavioural Pattern
- Purpose: Add functionality to a class without changing it
- Implementation:
  - Methods of class A get a visitor object as argument
  - The visitor's interface methods are called at specific points of the computation and have an A object as argument (at least)
  - This allows different additional computations or side effects with one class method of class A
  - The functionality is parameterized by the different visitor objects and classes, so to speak
- Especially useful with traversal methods of complex objects (like DFS or BFS)

DFS/BFS Visitors II

- Methods common for DFS/BFS visitor interface
  - `startNode(v,g)` : *v* is white in the outer loop
  - `discoverNode(v,g)` : *v* changes from white to gray
  - `finishNode(v,g)` : *v* changes from gray to black
  - `treeEdge(e,g)` : visit edge with white target node
- Methods specific to BFS visitor
  - `examineNode(v,g)` : *v* is taken off the queue
  - `grayTarget(e,g)` : gray target node
  - `blackTarget(e,g)` : black target node
- Methods specific to DFS visitor:
  `backEdge(e,g)`, `forwardEdge(e,g)`, `crossEdge(e,g)`

Topological Order

- Order the vertices such that if (n, m) is an edge, n comes before m
- Only exists for acyclic graphs
- Algorithm: sort vertices according to decreasing finishing times of DFS
- This can be easily implemented by a DFS visitor
- As each vertex is finished, add it to the front of a linked list
- The visitor contains this list as member variable
- After all vertices have been visited by DFS, the visitor holds the result
- Application example: A constraint graph that locally specifies which action must precede another

Topological Order Example