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# What is a Logic Translation?

**In memoriam Joseph Goguen**

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**Abstract.** We study logic translations from an abstract perspective, without any commitment to the structure of sentences and the nature of logical entailment, which also means that we cover both proof-theoretic and model-theoretic entailment. We show how logic translations induce notions of logical expressiveness, consistency strength and sublogic, leading to an explanation of paradoxes that have been described in the literature. Connectives and quantifiers, although not present in the definition of logic and logic translation, can be recovered by their abstract properties and are preserved and reflected by translations under suitable conditions.

## 1. Introduction

The development of a notion of *logic translation* presupposes the development of a notion of *logic*. Logic has been characterised as the study of sound reasoning, of what follows from what. Hence, the notion of logical consequence is central. It can be obtained in two complementary ways: via the model-theoretic notion of satisfaction in a model and via the proof-theoretic notion of derivation according to proof rules. These notions can be captured in a completely abstract manner, avoiding any particular commitment to the nature of models, sentences, or rules. The paper [40], which is the predecessor of our current work, explains the notion of logic along these lines.

Similarly, an abstract notion of logic translation can be developed, both at the model-theoretic and at the proof-theoretic level. A central question concerning such translations is their interaction with logical structure, such as given by logical connectives, and logical properties. Moreover, an abstract notion of logic translation opens the door to the abstract study of other notions like sublogic and expressiveness.

We shed light on the paradoxical situations spotted out by Béziau [4] and others, namely that a sublogic may turn out to be more expressive than its superlogic. For this, it is important to be precise about the notions of sublogic and of expressiveness. Expressiveness cannot be equated with consistency strength, rather, it corresponds to discriminatory strength. It turns out that discriminatory strength increases when fewer symbols are axiomatised in a specific way.

We also study logical consequence, satisfaction and translation in the context of a "signature", that is, a vocabulary containing the non-logical symbols, like propositional variables, relation symbols, function symbols, and so on. As with sentences, models and rules, we leave the nature of signatures completely open; this is achieved by the use of category theory and made explicit in the notion of *institution*. Using signatures, the ingredients of a logic (and also of a logic translation) become explicitly indexed by the context. This can be motivated philosophically by arguments like those given by Peirce [42] for his "interpretants", which allow for context dependency of denotation in his semiotics. Technically, this allows more complex concepts like quantification, interpolation, definability, etc., and their interaction with translations, to be studied at the same level of abstraction as indicated above.

Actually, both the model-theoretic and, even more, the signature-indexed view of logic seem to be underrepresented in the universal logic community. We therefore provide a motivation of and introduction to these aspects of logic, starting with the more widely used concept of entailment relation, then proceeding to satisfaction systems that capture model theory, and finally to the level where indexing by signatures is made explicit.

The main contribution of this paper is twofold: on the one hand, we develop several results concerning the interaction of different types of translations with different kinds logical connectives and quantifiers. On the other hand, we show that some well-known translations between (variants of) classical and intuitionistic logic can be turned into semantic translations (thereby also providing a simpler and more conceptual explanation of their faithfulness).

This paper was presented at the contest "How to translate a logic into another one?" of the 2nd World Congress and School on Universal Logic in Xi'an, China, in August 2007, where it won the contest prize.

*Joseph Goguen.* Joseph Goguen, with whom we co-authored the precursor paper "What is a logic?" [40], died on July 3rd, 2006. The scientific community lost a great scientist, well-known for his pioneering research in many diverse areas. We lost also a close friend and teacher. Shortly before his death, we were privileged to take part in the Festschrift colloquium for his 65th birthday [19]. His most important message to the participants was a commitment to solidarity and cooperation.

## 2. Entailment Relations

The most basic ingredient of a logic is an 'entailment' relation between sentences. In the literature this is sometimes called a 'consequence' relation. The notion of (abstract) consequence relation has been formalised by Gentzen, Tarski and Scott [22, 45, 2].[1]

**Definition 2.1.** *An* entailment relation *(abbreviated: ER) $\mathcal{S} = (S, \vdash)$ on a set of sentences $S$ is a binary relation $\vdash \subseteq \mathcal{P}(S) \times S$ between sets of sentences and sentences such that*

1. reflexivity: *for any $\varphi \in S$, $\{\varphi\} \vdash \varphi$,*
2. monotonicity: *if $\Gamma \vdash \varphi$ and $\Gamma' \supseteq \Gamma$ then $\Gamma' \vdash \varphi$,*
3. transitivity: *if $\Gamma \vdash \varphi_i$, for $i \in I$, and $\Gamma \cup \{\varphi_i \mid i \in I\} \vdash \psi$, then $\Gamma \vdash \psi$.*

*An ER is said to be* compact *when for each $E \vdash \varphi$ there exists a finite subset $E_0 \subseteq E$ such that $E_0 \vdash \varphi$.*

A generalisation by Avron [2] also allows for inclusion of substructural (e.g. non-monotonic) logics. For simplicity, we here restrict ourselves to Tarskian entailment relations, i.e. those satisfying the above axioms. However, note that substructural logics can be encoded into Tarskian entailment relations by considering whole sequents as sentences. Entailment between such sequents *is* monotone.

**Example 2.2.** Intuitionistic propositional logic (**IPL**) has sentences given by the following grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \rightarrow \varphi_2 \mid \top \mid \bot$$

where $p$ denotes propositional variables, taken from a countable supply. The entailment relation is the minimal relation satisfying the properties listed in Table 1.

Classical propositional logic (**CPL**) is defined like **IPL**, except that the property $\neg\neg\varphi \vdash \varphi$ is added.

The study of translations between entailment relations has a long history. The origins date back the 1920ies and 1930ies, when Kolmogorov, Glivenko, Gödel, Gentzen and others studied translations among **IPL**, **CPL** and propositional modal logic [32, 23, 24, 25, 21], see [43, 16, 10, 17] for surveys and discussion. In these works, "logic" is mostly identified with "entailment relation". The point of the present work is to take a broader view, which will be detailed below.

**Definition 2.3.** *Given two ERs $\mathcal{S}_1 = (S_1, \vdash^1)$ and $\mathcal{S}_2 = (S_2, \vdash^2)$, a* morphism of entailment relations *$\alpha \colon \mathcal{S}_1 \longrightarrow \mathcal{S}_2$, ER-morphism for short, is a function $\alpha \colon S_1 \longrightarrow S_2$ such that*

$$\Gamma \vdash^1 \varphi \text{ implies } \alpha(\Gamma) \vdash^2 \alpha(\varphi)$$

*If also the converse implication holds, the ER morphism is said to be* conservative.[2]

[1] More general notions work with multisets instead of sets of premises [2]. Here, we use the notion that seems to be most natural and simplest.

[2] Prawitz and Malmnäs [43] also use a more permissive notion of conservative translation where the equivalence is only required for $\Gamma = \emptyset$.

ER morphisms compose and there are obvious identity ER morphisms; hence, ERs and ER morphisms form a category $\mathbb{ER}$. See [17] for properties of this category and its subcategory of conservative morphisms.

**Example 2.4.** Kolmogorov's translation $K$ of classical propositional logic (**CPL**) into intuitionistic propositional logic (**IPL**) [32] adds a double-negation for each subsentence:

$$\begin{array}{l}
K(p) = \neg\neg p \quad \text{for each propositional symbol} \quad p \\
K(\neg\varphi) = \neg K(\varphi) \\
K(\varphi_1 \wedge \varphi_2) = \neg\neg(K(\varphi_1) \wedge K(\varphi_2)) \\
K(\varphi_1 \vee \varphi_2) = \neg\neg(K(\varphi_1) \vee K(\varphi_2)) \\
K(\varphi_1 \rightarrow \varphi_2) = \neg\neg(K(\varphi_1) \rightarrow K(\varphi_2)) \\
K(\top) = \top \\
K(\bot) = \bot
\end{array}$$

This is a conservative ER morphism [43]. Below we will develop our own proof of this result (see Cor. 3.13).

ERs can be alternatively described as closure operators [8, 12]:

**Definition 2.5.** *A* closure operator *on a set $S$ is a map $C\colon \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$ such that for $\Gamma, \Delta \subseteq S$,*

- $\Gamma \subseteq C(\Gamma)$
- $\Gamma \subseteq \Delta$ *implies* $C(\Gamma) \subseteq C(\Delta)$
- $C(C(\Gamma)) \subseteq C(\Gamma)$.

*Given closure operators $(S_1, C_1)$ and $(S_2, C_2)$, a function $\alpha\colon S_1 \longrightarrow S_2$ is said to be* continuous, *if $\alpha(C_1(\Gamma)) \subseteq C_2(\alpha(\Gamma))$ for any $\Gamma \subseteq S_1$.*

**Proposition 2.6.** $\mathbb{ER}$ *is isomorphic to the category of closure operators and continuous functions.*

*Proof.* Given an ER, define a closure operator by $C(\Gamma) = \{\varphi \mid \Gamma \vdash \varphi\}$. Conversely, given a closure operator, obtain an ER by defining $\Gamma \vdash \varphi$ iff $\varphi \in C(\Gamma)$. Morphisms are left untouched. This is easily shown to be the desired isomorphism. □

### 2.1. Logical connectives

A common requirement on translations (e.g. adopted by Wojcicki [48] and made explicit by Epstein and Krajewski [16]) is that they are *schematic* or *grammatical* [16], which means that they preserve the algebraic structure of the sentences — in other words, they are defined inductively over the structure of the sentences in the source ER. However, this preservation requirement rules out e.g. the standard translation of modal logic to first-order logic, which adds a quantifier at the very top. [16] lists other translations that are not schematic. Moreover, since we consider abstract ERs, it is not clear from the outset what the algebraic structure of the sentences is.

Hence, rather than treating logical connectives as given by an algebraic structure that must be preserved, it seems to be more promising to *characterise* logical

| connective | defining property |
|---|---|
| proof-theoretic conjunction $\wedge$ | $\Gamma \vdash \varphi \wedge \psi$ iff $\Gamma \vdash \varphi$ and $\Gamma \vdash \psi$ |
| proof-theoretic disjunction $\vee$ | $\varphi \vee \psi, \Gamma \vdash \chi$ iff $\varphi, \Gamma \vdash \chi$ and $\psi, \Gamma \vdash \chi$ |
| proof-theoretic implication $\rightarrow$ | $\Gamma, \varphi \vdash \psi$ iff $\Gamma \vdash \varphi \rightarrow \psi$ |
| proof-theoretic truth $\top$ | $\Gamma \vdash \top$ |
| proof-theoretic falsum $\bot$ | $\bot \vdash \varphi$ |
| proof-theoretic negation $\neg$ | $\Gamma, \varphi \vdash \bot$ iff $\Gamma \vdash \neg\varphi$ |

TABLE 1. Properties of proof-theoretic connectives

connectives *in terms of their properties*. This is well-known in proof theory; we adapt the standard definitions from [34], see also [40]. Note that our definition of negation also covers intuitionistic negation (unlike the treatment in [2], which assumes $\neg\neg$-elimination).

These properties characterise connectives essentially by their *proof-theoretic* behaviour; they mostly directly correspond to proof rules. Below, we will also introduce *semantic* connectives. An ER is said to *have a proof-theoretic connective* (or that the connective is present in it) if it is possible to define a corresponding operation on sentences with the properties specified in Table 1. For example, both **IPL** and **CPL** have all proof-theoretic connectives. We will see below that, by contrast, **IPL** has only rather few model-theoretic connectives.

Thus prepared, we can recover Wojcicki's notion of schematic translation at the level of abstract ERs as follows:

**Definition 2.7.** *Given an ER $\mathcal{S} = (S,\vdash)$, let $\Sigma_{\mathcal{S}}$ be the (single-sorted) algebraic signature that has the elements of $S$, i.e. all sentences of $\mathcal{S}$, as constants and all the proof-theoretic connectives present in $\mathcal{S}$ as operation symbols (where each operation symbol inherits its arity from the connective). By interpreting sentences as themselves and operation symbols as the corresponding proof-theoretic connectives, we obtain a $\Sigma_{\mathcal{S}}$-algebra with carrier set $S$, which by abuse of notation we will also denote by $\mathcal{S}$.*

*Fix a set $X = \{X_1, X_2, \ldots\}$ of (schematic) variables. A* propositionally schematic sentence *is a term $\Phi \in T_{\Sigma_{\mathcal{S}}}(X)$ over $\Sigma_{\mathcal{S}}$ with variables from $X$. Any valuation $\nu\colon X \longrightarrow S$ is a substitution of sentences for schematic variables; it can be uniquely extended to $\nu^{\#}\colon T_{\Sigma_{\mathcal{S}}}(X) \longrightarrow S$. Let $\Phi[\nu] = \nu^{\#}(\Phi)$ denote the result of applying substitution $\nu$ to schematic sentence $\Phi$.*

*An ER morphism $\alpha\colon \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ is called* propositionally schematic *if for each $n$-ary connective $c$ present in $\mathcal{S}_1$, there is a propositionally schematic sentence $\Phi_c$ in schematic variables $X_1, \ldots, X_n$ in $\mathcal{S}_2$ such that for all sentences $\varphi_1, \ldots, \varphi_n \in S_1$*

$$\alpha(c(\varphi_1, \ldots, \varphi_n)) \dashv\vdash \Phi_c[\alpha(\varphi_1)/X_1, \ldots, \alpha(\varphi_n)/X_n]$$

Here, $\dashv\vdash$ means entailment in both directions. Most of the ER morphisms in this paper will be propositionally schematic, but recall that there are important exceptions. The above definition does not require sentences to be inductively defined; they could also be defined in a non-well-founded way.

We will now turn to preservation properties.

**Definition 2.8.** *An ER-morphism $\alpha\colon \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ is said to* transport *a connective if the presence of the connective in $\mathcal{S}_1$ implies its presence in $\mathcal{S}_2$. The converse implication is called* reflection *of connectives. If two ERs have a connective c, the property that $\alpha(c(\varphi_1,\dots,\varphi_n)) \dashv\vdash c(\alpha(\varphi_1),\dots,\alpha(\varphi_n))$ is called* preservation *of the connective.*

It is straightforward to prove the following:

**Proposition 2.9.** *Let $\alpha\colon \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ be a conservative ER-morphism.*

1. *It reflects all those proof-theoretic connectives under which $\alpha(S_1)$ is closed.*
2. *If in addition $\alpha$ is also surjective, then it transports and reflects all proof-theoretic connectives, and preserves all existing proof-theoretic connectives.*

### 2.2. Consistency Strength

A theory $\Gamma \subseteq S$ is *consistent* if $\Gamma \not\vdash \varphi$ for some $\varphi$, otherwise, it is inconsistent.[3] An ER is consistent if there is at least one consistent theory. Perhaps the most famous inconsistent logic is Frege's *Begriffsschrift*, a higher-order logic amenable to Russell's paradox.

The notion of consistency strength is usually formulated as follows:

**Definition 2.10.** *If the consistency of a theory $\Gamma_1$ (or an ER $\mathcal{S}_1$) implies the consistency of a theory $\Gamma_2$ (or an ER $\mathcal{S}_2$), the former is said to have* consistency strength *greater or equal to that of the latter. Formally, we denote this by $\Gamma_1 \succeq \Gamma_2$ (or $\mathcal{S}_1 \succeq \mathcal{S}_2$).*

**Example 2.11.** By Gödel's famous result, the first-order theories ZF and ZFC have equal consistency strength [26].

An ER morphism can transport consistency strength in both directions, depending on suitable conditions (generalising results in [17]):

**Proposition 2.12.** *Let $\alpha\colon \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ be an ER morphism.*

1. *Let $\alpha$ be conservative. Then it transports consistency, i.e. for any theory $\Gamma$, we have $\Gamma \succeq \alpha(\Gamma)$, and $\mathcal{S}_1 \succeq \mathcal{S}_2$.*
2. *Let $\alpha$ be surjective. Then it reflects consistency, i.e. $\Gamma \preceq \alpha(\Gamma)$, and $\mathcal{S}_1 \preceq \mathcal{S}_2$.*
3. *Let $\alpha$ preserve proof-theoretic falsum (which assumes that both $\mathcal{S}_1$ and $\mathcal{S}_2$ have proof-theoretic falsum). Then $\Gamma \preceq \alpha(\Gamma)$, and $\mathcal{S}_1 \preceq \mathcal{S}_2$.*

*Proof.* 1. and 2. are straightforward. For 3., note that in presence of proof-theoretic falsum, consistency of $\Gamma$ is equivalent to $\Gamma \not\vdash \bot$, and consistency of an ER is equivalent to $\emptyset \not\vdash \bot$. $\square$

[3] Inconsistent theories are called *trivial* in [10].

A non-conservative, non-surjective ER morphism (even if it is an inclusion) need neither transport nor reflect consistency, as the following counterexamples show:

**Example 2.13.** Consider classical propositional logic **CPL** (Ex. 2.2), which has an explicit truth, and let **T** be the restriction of **CPL** to truth as the only sentence. Then **T** is trivially inconsistent, but it has a (non-surjective) inclusion[4] into **CPL**, which is consistent. This shows that such inclusions do not need to reflect consistency. They do not need to preserve consistency either: consider the inclusion of **CPL** into the logic which has sentences like **CPL**, but an entailment relation that is the universal relation.[5]

### 2.3. Expressiveness

Consistency strength should not be equated with expressiveness, as Humberstone [30, 31] argues convincingly. Indeed, Humberstone shows that under rather mild conditions, increasing logical strength coincides with decreasing discriminatory strength (and the latter is closely related to expressiveness). Unfortunately, we cannot adopt Humberstone's notion of discriminatory strength here, because it is based on a notion of substitution, which we cannot expect to be available for arbitrary ERs (for example, it could very well make sense to consider ERs with graphs as sentences). Instead, we take a different measure of expressiveness: the more logics are embeddable into a logic, the more expressive it is. This leads to the following definition (cf. also [33]):

**Definition 2.14.** $\mathcal{S}_1 \leq^{ER} \mathcal{S}_2$ *iff there is some conservative* $\alpha\colon \mathcal{S}_1 \longrightarrow \mathcal{S}_2$.

$\mathcal{S}_1 \leq^{ER} \mathcal{S}_2$ is read as "$\mathcal{S}_1$ is at most as expressive as $\mathcal{S}_2$" or "$\mathcal{S}_2$ is at least as expressive as $\mathcal{S}_1$".

For example, Kolmogorov's double negation translation (Ex. 2.4) shows that **CPL** $\leq^{ER}$ **IPL**.

Note that $\leq^{ER}$ is only a pre-order, not a partial order. $\mathcal{S}_1$ and $\mathcal{S}_2$ are *equally expressive* ($\mathcal{S}_1 \equiv^{ER} \mathcal{S}_2$), if both $\mathcal{S}_1 \leq^{ER} \mathcal{S}_2$ and $\mathcal{S}_2 \leq^{ER} \mathcal{S}_1$. $\mathcal{S}_1$ is *strictly less expressive* than $\mathcal{S}_2$ ($\mathcal{S}_1 <^{ER} \mathcal{S}_2$), if $\mathcal{S}_1 \leq^{ER} \mathcal{S}_2$ but not $\mathcal{S}_1 \equiv^{ER} \mathcal{S}_2$.

It might contradict one's intuitions that a logic with a stronger axiomatisation is generally *less* expressive than a logic with a weaker axiomatisation. For example, Humberstone [30] shows that the modal logic **KT** is strictly less expressive than **K**. Note though that this is based on a stronger notion of ER morphism than we use: namely, propositional connectives have to be preserved — without this requirement, it is not clear whether **KT** is *strictly* less expressive than **K**. A similar example arises when comparing $\mathbf{PDL}^{\models}$ generated by the *semantic* entailment of basic propositional dynamic logic [5]. We now briefly introduce this logic.

[4] If we want to take models into account, we can let them be the same for both logics and define the inclusion to leave them untouched.

[5] The universal relation contains *all* pairs of elements. When taking models into account, we can use the empty model class for this logic.

**Definition 2.15.** *The sentences of propositional dynamic logic* $\mathbf{PDL}^{\models}$ *are generated by the grammar for* **CPL***, extended by*

$$\varphi ::= \ldots \mid [\pi]\varphi$$

*where programs* $\pi$ *are generated from a set* $P$ *of atomic programs by*

$$\pi ::= P \mid \pi_1 \cup \pi_2 \mid \pi_1 ; \pi_2 \mid \pi^*$$

*A regular Kripke model* $(W, V, R)$ *for* $\mathbf{PDL}^{\models}$ *consists of a set of worlds* $W$*, a map* $V$ *from propositional variables to subsets of* $W$*, and a family* $(R_\pi)$ *indexed by programs* $\pi$*, such that each* $R_\pi$ *is a binary transition relation on* $W$*, subject to the following regularity conditions:*

- $R_{\pi_1 \cup \pi_2} = R_{\pi_1} \cup R_{\pi_2}$,
- $R_{\pi_1 ; \pi_2} = R_{\pi_1} \circ R_{\pi_2}$,
- $R_{\pi^*} = (R_\pi)^*$, *the reflexive-transitive closure of* $R_\pi$.

*Satisfaction of a sentence in a world* $w$ *is defined inductively:*

- $w \Vdash p$ *if* $w \in V(p)$,
- $w \Vdash [\pi]\varphi$ *if for all* $v$ *with* $wR_\pi v$, $v \Vdash \varphi$
- *propositional connectives are handled as usual.*

*Define an ER out of this by letting* $\Gamma \vdash \varphi$ *if for all models* $(W, V, R)$ *and all worlds* $w \in W$*, if* $w \Vdash \gamma$ *for all* $\gamma \in \Gamma$*, then* $w \Vdash \varphi$.

**Definition 2.16.** $\mathbf{PDL}^{\models}$-*Triv extends* $\mathbf{PDL}^{\models}$ *by adding, for any program* $\pi$ *and sentence* $\varphi$*, the axiom*

$$[\pi]\varphi \leftrightarrow \varphi$$

Equivalently, $\mathbf{PDL}^{\models}$-*Triv* may be obtained by restricting attention in $\mathbf{PDL}^{\models}$ to those models where all transition relations are identities and all propositions are interpreted in the same way in all worlds, which means that $\mathbf{PDL}^{\models}$-*Triv* is essentially the same logic as **CPL**.

**Proposition 2.17.** $\mathbf{PDL}^{\models}$ *is strictly more expressive than* $\mathbf{PDL}^{\models}$-*Triv:* $\mathbf{PDL}^{\models} >^{ER} \mathbf{PDL}^{\models}$-*Triv. (By the above remark, thus also* $\mathbf{PDL}^{\models} >^{ER}$ **CPL***.)*

*Proof.* $\mathbf{PDL}^{\models}$-*Triv* can be conservatively mapped to $\mathbf{PDL}^{\models}$ by erasing all modalities. However, there is no conservative ER morphism in the converse direction: suppose there were one, say, $\alpha : \mathbf{PDL}^{\models} \to \mathbf{PDL}^{\models}$-*Triv*. Since $\mathbf{PDL}^{\models}$ is not compact [5], there is an infinite set of sentences $\Gamma$ (w.l.o.g. assumed to be pairwise logically inequivalent) and a sentence $\varphi$ such that $\Gamma \vdash \varphi$, but for no finite subset $\Gamma' \subseteq \Gamma$, $\Gamma' \vdash \varphi$. This situation translates along $\alpha$, contradicting compactness of $\mathbf{PDL}^{\models}$-*Triv*. □

Comparing $\mathbf{PDL}^{\models}$ and $\mathbf{PDL}^{\models}$-*Triv*, we note:

- $\mathbf{PDL}^{\models}$ (obviously) has a weaker axiomatisation than $\mathbf{PDL}^{\models}$-*Triv*.
- $\mathbf{PDL}^{\models}$ (obviously) has less consistency strength than $\mathbf{PDL}^{\models}$-*Triv*, i.e. $\mathbf{PDL}^{\models} \preceq \mathbf{PDL}^{\models}$-*Triv*.
- $\mathbf{PDL}^{\models}$ is strictly more expressive than $\mathbf{PDL}^{\models}$-*Triv*.

This is in accordance with Humberstone's [31] observation that weaker logics generally have more flexibility in interpretation of symbols, and therefore more *discriminatory strength*, that is, an increased ability to distinguish between different concepts — and this means more expressiveness. Here, in $\mathbf{PDL}^{\models}$-$Triv$ we are forced to identify $[\pi]\varphi$ with $\varphi$, so that the modalities do not introduce anything new. Hence, a stronger axiomatisation may constrain the symbols so much that fewer things can be expressed.

Indeed, this contravariance between expressiveness and consistency strength can be made formal: by Prop. 2.12 we immediately have

**Proposition 2.18.** *$\mathcal{S}_1 \geq^{ER} \mathcal{S}_2$ implies $\mathcal{S}_1 \preceq \mathcal{S}_2$.*

However, we cannot infer the converse, that logics with weaker axiomatisation are automatically more expressive. Humberstone [30] has the example of the modal logics **L** and **K**. **L** has a weaker axiomatisation than **K**, which means that there is a (non-conservative) ER morphism $\mathbf{L} \to \mathbf{K}$, but this does not imply the existence of a conservative morphism in the other direction.[6] This shows that moving to a weaker axiomatisation not always leads to preservation or increase of discriminatory strength and expressiveness.

Let us now go to one extreme. Inconsistent logics are inexpressive, because they do not have any discriminatory strength at all:

**Proposition 2.19.** *If $\mathcal{S}_1$ is inconsistent and $\mathcal{S}_2$ has a proof-theoretic truth, then $\mathcal{S}_1 \leq^{ER} \mathcal{S}_2$.*

*Proof.* Map any sentence in $\mathcal{S}_1$ to $\top$ in $\mathcal{S}_2$. □

How does expressiveness interact with the presence of logical connectives? The extra conditions needed in Prop. 2.9 make it no surprise that a more expressive ER may have fewer connectives then a less expressive one: consider e.g. propositional logic with just conjunction embedded into a variant of Horn logic consisting of implications between conjunctions — the former logic has proof-theoretic conjunction, while the latter has not. Another example that we encountered is many-sorted first-order logic embedded into its extension with second-order induction axioms [41] — the latter are not closed under standard unary and binary logical connectives at all.

In general, we cannot expect that equally expressive ERs have the same connectives. This only holds under additional requirements, namely those of Prop. 2.9. Indeed, *equivalent* ERs have the same connectives. The notion of equivalence on ERs is obtained by specialising the more general notion for logics in [40], the notion is very similar to equipollence in [9] or to a more specialised notion (corresponding to schematic translations) used in the study of algebraisation of logics [6].

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[6] Again note that Humberstone uses a stronger notion of ER morphism. Hence, it is unclear whether there is a conservative ER morphism from **K** to **L**.

**Definition 2.20.** *Two ERs $\mathcal{S}_1$ and $\mathcal{S}_2$ are* equivalent*, if there are conservative maps $\alpha_1\colon \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ and $\alpha_2\colon \mathcal{S}_2 \longrightarrow \mathcal{S}_1$ such that $\varphi \dashv\vdash \alpha_2(\alpha_1(\varphi))$ and $\psi \dashv\vdash \alpha_1(\alpha_2(\psi))$. (Here, $\dashv\vdash$ means entailment in both directions.)*

**Proposition 2.21.** *Equivalent ERs have the same proof-theoretic connectives.*

*Proof.* Translate the constituent sentences to the other entailment relation, apply the connective there, and then translate it back. $\square$

### 2.4. Theoroidal morphisms

When *encoding* an ER into another one, e.g. for the purpose of re-using a theorem prover, we can work with a more general definition of ER morphism, allowing the target sentences to be considered in the context of a base theory.

**Definition 2.22.** *Given two ERs $\mathcal{S}_1 = (S_1, \vdash^1)$ and $\mathcal{S}_2 = (S_2, \vdash^2)$, a* simple theoroidal [29] ER morphism *$(\alpha, \Delta)\colon \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ is a function $\alpha\colon S_1 \longrightarrow S_2$ together with a theory $\Delta \subseteq S_2$ such that*

$$\Gamma \vdash^1 \varphi \text{ implies } \Delta \cup \alpha(\Gamma) \vdash^2 \alpha(\varphi)$$

See [29, 38, 39] for further motivation and examples. Simple theoroidal morphisms are an abbreviation rather than a new concept since they are special cases of ordinary morphisms $\mathcal{S}_1 \to \mathcal{S}_2^\Delta$, where $\mathcal{S}_2^\Delta = (S_2, \vdash^\Delta)$ is given by $\Gamma \vdash^\Delta \varphi$ iff $\Gamma \cup \Delta \vdash^2 \varphi$. (Ordinary ER morphisms are sometimes called *plain* in order to stress the distinction from simple theoroidal ones.)

The definitions of conservative morphism and expressiveness carry over. For the latter, we use the notation $\leq^{ER\text{-}th}$. Before providing an example using simple theoroidal ER morphisms, we define propositional Horn Clause Logic:

**Definition 2.23.** *Propositional Horn Clause Logic* **PHCL** *has sentences*

$$p_1 \wedge \cdots \wedge p_n \to p$$

*where $p, p_1, \ldots, p_n$ are propositional variables (taken from a countable supply). (In the case of $n = 0$, the sentence is written as $p$.) Entailment is generated by the following rules:*

$$(\text{refl}) \quad \frac{}{p_1 \wedge \cdots \wedge p_n \to p_i}$$

$$(\text{weak-ctr}) \quad \frac{p_1 \wedge \cdots \wedge p_n \to p \quad \text{and} \quad \{p_1, \ldots, p_n\} \subseteq \{q_1, \ldots, q_m\}}{q_1 \wedge \cdots \wedge q_m \to p}$$

$$(\text{cut}) \quad \frac{p_1 \wedge \cdots \wedge p_n \to q_i \quad \text{and} \quad q_1 \wedge \cdots \wedge q_m \to q}{p_1 \cdots \wedge p_n \wedge q_1 \wedge \cdots \wedge q_{i-1} \wedge q_{i+1} \wedge \cdots \wedge q_m \to q}$$

**Proposition 2.24.** *The entailment relation of* **PHCL** *enjoys a deduction theorem: if*

$$\Gamma \cup \{q_1, \ldots, q_m\} \vdash p_1 \wedge \cdots \wedge p_n \to p,$$

*then*

$$\Gamma \vdash q_1 \wedge \ldots \wedge q_m \wedge p_1 \wedge \cdots \wedge p_n \to p.$$

*(where $p, p_1, \ldots, p_n, q_1, \ldots, q_m$ are propositional variables).*

*Proof.* By induction over $m$, we can reduce the proof to the case $m = 1$. By induction over the derivations, we transform any derivation of $\Gamma \cup \{q_1\} \vdash p_1 \wedge \cdots \wedge p_n \rightarrow p$ to a derivation of $\Gamma \vdash q_1 \wedge p_1 \wedge \cdots \wedge p_n \rightarrow p$. Every use of the premise $q_1$ in the derivation is replaced with $q_1 \rightarrow q_1$ obtained by (refl). Rule (weak-ctr) can be used to handle the cases of multiple or no uses of $q_1$. $\square$

Call an ER countable if its set of sentences is countable.

**Proposition 2.25.** *The entailment relation of* **PHCL** *has maximal expressiveness among compact countable ERs (when admitting simple theoroidal ER morphisms).*

*Proof.* Let $\mathcal{S}$ be any compact ER. We want to construct a conservative $(\alpha, \Delta) \colon \mathcal{S} \longrightarrow$ **PHCL**. For each sentence $\varphi$, we introduce a propositional variable which we denote by $\alpha(\varphi)$. $\Delta$ consist of all sentences $\alpha(\varphi_1) \wedge \ldots \wedge \alpha(\varphi_n) \rightarrow \alpha(\varphi)$ such that $\{\varphi_1, \ldots, \varphi_n\} \vdash \varphi$ in $\mathcal{S}$. Using the properties of ERs, it is easy to show that $\Delta$ is closed under rules (refl), (weak-ctr) and (cut) above. Hence, $\Delta$ is closed under entailment, i.e., if $\Delta \vdash \gamma$ in **PHCL** then $\gamma \in \Delta$.

We first show that $(\alpha, \Delta)$ is a simple theoroidal ER morphism. Let $\Gamma \vdash \varphi$ in $\mathcal{S}$. By compactness there exists $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma$ such that $\{\varphi_1, \ldots, \varphi_n\} \vdash \varphi$. Hence $\alpha(\varphi_1) \wedge \ldots \wedge \alpha(\varphi_n) \rightarrow \alpha(\varphi) \in \Delta$ and since $\{\alpha(\varphi_1) \wedge \ldots \wedge \alpha(\varphi_n) \rightarrow \alpha(\varphi), \alpha(\varphi_1), \ldots, \alpha(\varphi_n)\} \vdash \alpha(\varphi)$ in **PHCL** it follows that $\Delta \cup \alpha(\Gamma) \vdash \alpha(\varphi)$.

Next we show that $(\alpha, \Delta)$ is conservative. Let $\Delta \cup \alpha(\Gamma) \vdash \alpha(\varphi)$. By compactness of the entailment relation of **PHCL**, there exists $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma$ such that $\Delta \cup \{\alpha(\varphi_1), \ldots, \alpha(\varphi_n)\} \vdash \alpha(\varphi)$. By Prop. 2.24, we obtain that $\Delta \vdash \alpha(\varphi_1) \wedge \ldots \wedge \alpha(\varphi_n) \rightarrow \alpha(\varphi)$. Since $\Delta$ is closed under entailment, we get $\alpha(\varphi_1) \wedge \ldots \wedge \alpha(\varphi_n) \rightarrow \alpha(\varphi) \in \Delta$ which implies that $\{\varphi_1, \ldots, \varphi_n\} \vdash \varphi$ and hence $\Gamma \vdash \varphi$.

Thus $(\alpha, \Delta)$ is a conservative simple theoroidal ER morphism, showing that $\mathcal{S} \leq^{ER\text{-}th}$ **PHCL**. $\square$

**Corollary 2.26.** *Let* $\mathbf{CPL}_\omega$ *be* **CPL** *with infinitary countable conjunctions (with* $\vdash$ *generated by the rules for* **CPL** *plus the obvious infinitary rules for conjunction). Every compact countable ER can be conservatively embedded into* $\mathbf{CPL}_\omega$*, using a plain ER morphism.*

*Proof.* **PHCL** obviously can be conservatively embedded into **CPL**, and this in turn into $\mathbf{CPL}_\omega$. The results then follows by using the translation of Prop. 2.25, followed by $\varphi \mapsto \varphi \wedge \bigwedge \Delta$ (where $\bigwedge \Delta$ is the conjunction of all elements of $\Delta$). $\square$

## 3. Model Theory

While entailment relations capture entailment, the so-called 'rooms' (in the terminology of [27]) capture the Tarskian notion of satisfaction of a sentence in a model:

**Definition 3.1.** *A* room $\mathcal{R} = (S, \mathcal{M}, \models)$ *consists of*

- *a set of* $S$ *of* sentences*,*
- *a class*[7] $\mathcal{M}$ *of* models*, and*
- *a binary relation* $\models \subseteq \mathcal{M} \times S$*, called the* satisfaction relation*.*

A theory $\Gamma \subseteq S$ is *satisfiable*, if it has a model $M$ (i.e., a model $M \in \mathcal{M}$ such that $M \models \varphi$ for $\varphi \in \Gamma$). *Semantic entailment* in a room is defined as usual: for $\Gamma \subseteq S$ and $\varphi \in S$, we write $\Gamma \models \varphi$, if all models satisfying all sentences in $\Gamma$ also satisfy $\varphi$.

The following result is folklore [2]:

**Proposition 3.2.** *Semantic entailment* $\models$ *is a Tarskian entailment relation.*

**Definition 3.3.** *A* logic room $(S, \mathcal{M}, \models, \vdash)$ *consists of a room* $(S, \mathcal{M}, \models)$ *and an entailment relation* $(S, \vdash)$*. A logic room is* sound*, if*

$$\Gamma \vdash \varphi \textit{ implies } \Gamma \models \varphi,$$

*and is* complete*, if the converse holds. It is* weakly complete*, if* $\emptyset \models \varphi$ *implies* $\emptyset \vdash \varphi$*.*

Tarskian semantics gains much of its importance from the fact that it allows definitions of ERs that are often easier to grasp and closer to intuitive concepts than definitions of ERs using proof rules. For example, we can equip the entailment relation **CPL** (classical propositional logic) with a semantics by taking valuations of the propositional variables into $\{T, F\}$ as models.[8] Then, the usual sentences built by Boolean connectives (let us here assume that the only connectives are implication and negation) can be given the standard truth-table semantics, leading to a room, which — by abuse of notation — we also write as **CPL**. There is also an alternative semantics for the ER **CPL** using valuations into a Boolean algebra; we denote the resulting room by **CPL-BA**.[9] Likewise, the ER **IPL** has two different semantics that have become standard: **IPL-HA** uses valuations into Heyting algebras, whereas **IPL-K** uses Kripke models.

Let **WPL** denote a variant of classical logic [4], where the sentences are the same as in **CPL**, but the models are valuations of *all* sentences that respect the usual truth table semantics of conjunction, disjunction, and implication, but only *one half* of the condition for negations:

$M(\varphi \wedge \psi) = T$ iff $(M(\varphi) = T$ and $M(\psi) = T)$
$M(\varphi \vee \psi) = T$ iff $(M(\varphi) = T$ or $M(\psi) = T)$
$M(\varphi \to \psi) = T$ iff $(M(\varphi) = F$ or $M(\psi) = T)$
$M(\neg\varphi) = F$ if $M(\varphi) = T$

Using these different semantics of the same language, we now get two different Tarskian entailment relations, $\models^{\mathbf{CPL}}$ and $\models^{\mathbf{WPL}}$. In order to relate these semantically, we need to connect rooms. This is done via corridors:

[7]If we want to take model morphisms into account, a *category* of models should be used here. However, this would make no essential difference for the developments in this paper.
[8]A (unique) model morphism between two models exists if the values of variables change only from $F$ to $T$, but not vice versa.
[9]For first-order logic, this is called "Boolean-valued models".

**Definition 3.4.** *A* corridor $(\alpha, \beta)\colon (S_1, \mathcal{M}_1, \models_1) \longrightarrow (S_2, \mathcal{M}_2, \models_2)$ *consists of*

- *a sentence translation function* $\alpha\colon S_1 \longrightarrow S_2$*, and*
- *a model reduction function*[10] $\beta\colon \mathcal{M}_2 \longrightarrow \mathcal{M}_1$*, such that*

$$M_2 \models_2 \alpha(\varphi_1) \textit{ if and only if } \beta(M_2) \models_1 \varphi_1$$

*holds for each* $M_2 \in \mathcal{M}_2$ *and each* $\varphi_1 \in S_1$ *(*satisfaction condition*).*

Since corridors compose and there are obvious identity corridors, rooms and corridors form a category $\mathbb{R}oom$.

If $\beta(M_2) = M_1$, $M_1$ is called the *β-reduct* of $M_2$, and $M_2$ is called a *β-expansion* of $M_1$.

A *simple theoroidal* corridor $(\alpha, \beta, \Delta)$ is defined by analogy with a simple theoroidal ER morphism — $\Delta$ is a set of target sentences and $\beta$ needs to be defined only for those models that satisfy $\Delta$. Simple theoroidal corridors are an abbreviation rather than a new concept, since they are just special cases of ordinary corridors.[11]

**Proposition 3.5.** *The following are corridors:*

1. *a corridor from* **WPL** *to* **CPL***. For sentences, it is the identity. A* **CPL***-model is translated to a* **WPL***-model by extending the valuation of the propositional variables to all sentences in the classical way.*
2. *a corridor from* **CPL** *to* **WPL***. Sentence are mapped inductively:*
   - $\alpha(p) = p$*, for propositional variables* $p$*,*
   - $\alpha(\neg\varphi) = \alpha(\varphi) \to \neg\alpha(\varphi)$*,*
   - $\alpha(\varphi \to \psi) = \alpha(\varphi) \to \alpha(\psi)$*,* $\alpha(\varphi \wedge \psi) = \alpha(\varphi) \wedge \alpha(\psi)$*, and* $\alpha(\varphi \vee \psi) = \alpha(\varphi) \vee \alpha(\psi)$*.*

   *Models are translated by retaining the interpretation of all propositional variables and forgetting the rest.*

*Proof.* 1. Let $(\mathrm{id}, \beta)$ be the above defined corridor **WPL** → **CPL**. For any sentence $\varphi$ and any model $M$, by the definition of the satisfaction relation in **CPL**, we have that $M \models^{\mathbf{CPL}} \varphi$ if and only if $\beta(M)(\varphi) = T$. But $\beta(M)(\varphi) = T$ means precisely $\beta(M) \models^{\mathbf{WPL}} \varphi$. Hence we have the satisfaction condition for the corridor $(\mathrm{id}, \beta)$.

2. Let $(\alpha, \beta)$ denote the corridor **CPL** → **WPL** defined in the statement of the proposition. We prove by induction on the structure of the sentence $\varphi$ that for any **WPL**-model $M$ we have

$$M \models^{\mathbf{WPL}} \alpha(\varphi) \text{ if and only if } \beta(M) \models^{\mathbf{CPL}} \varphi$$

The base case, when $\varphi$ is a propositional variable, follows immediately from the definition of $\beta(M)$. The induction steps for $\to$, $\wedge$, and $\vee$ follow immediately by the induction hypothesis and because $\alpha$ preserves these connectives. The really interesting induction step is for $\neg$.

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[10] If model morphisms are taken into account, then a functor.

[11] Indeed, a simple theoroidal corridor $(\alpha, \beta, \Delta)$ : $(S, \mathcal{M}, \models) \to (S, \mathcal{M}', \models')$ is an ordinary corridor $(S, \mathcal{M}, \models) \to (S, (\mathcal{M}')^{\Delta}, (\models')^{\Delta})$, where $(\mathcal{M}')^{\Delta} = \{M \in \mathcal{M}' \mid M \models' \Delta\}$ and $(\models')^{\Delta})$ is the restriction of $\models'$.

Assume that $M \models^{\mathbf{WPL}} \alpha(\varphi)$ iff $\beta(M) \models^{\mathbf{CPL}} \varphi$. Then because of the semantic definition of $\neg$ in **CPL** we have that $\beta(M) \models^{\mathbf{CPL}} \neg\varphi$ iff $\beta(M) \not\models^{\mathbf{CPL}} \varphi$. By the induction hypothesis we have that $\beta(M) \not\models^{\mathbf{CPL}} \varphi$ iff $M \not\models^{\mathbf{WPL}} \alpha(\varphi)$ which means $M(\alpha(\varphi)) = F$. Now the conclusion follows by noticing that $M(\alpha(\varphi)) = F$ is equivalent to $M(\alpha(\varphi) \to \neg\alpha(\varphi)) = T$. The implication from the left to the right is immediate. For the converse implication it is enough to notice that if $M(\alpha(\varphi)) = T$ then $M(\alpha(\varphi) \to \neg\alpha(\varphi)) = F$. □

**Definition 3.6.** *A corridor is* model-expansive *if its model translation is surjective.*

The crucial difference between the two above corridors is that the second one is model-expansive, while the first one is not.

**Proposition 3.7.** *A corridor leads to an ER morphism between the induced semantic entailment relations of the rooms. If the corridor is model-expansive, then the ER morphism is conservative.*

*Proof.* Consider a corridor $(\alpha, \beta)\colon\ (S_1, \mathcal{M}_1, \models^1) \to (S_2, \mathcal{M}_2, \models^2)$. We have to prove that $\Gamma \models^1 \varphi$ implies $\alpha(\Gamma) \models^2 \alpha(\varphi)$. For this consider a model $M_2 \in \mathcal{M}_2$ such that $M_2 \models^2 \alpha(\Gamma)$. By the satisfaction condition for the corridor we have that $\beta(M_2) \models^1 \Gamma$ and by the hypothesis that $\beta(M_2) \models^1 \varphi$. By the satisfaction condition for the corridor, this time in the other direction, we have that $M_2 \models^2 \alpha(\varphi)$, which completes the proof of the ER morphism property.

Now let us assume that $(\alpha, \beta)$ is model expansive. We prove that the induced ER morphism is conservative. Suppose $\alpha(\Gamma) \models^2 \alpha(\varphi)$. We have to show that $\Gamma \models^1 \varphi$. Let $M_1 \in \mathcal{M}_1$ such that $M_1 \models \Gamma$. By the model expansive property for $(\alpha, \beta)$, there exists $M_2 \in \mathcal{M}_2$ such that $M_1 = \beta(M_2)$. By the satisfaction condition $M_2 \models^2 \alpha(\Gamma)$ and, by the hypothesis, $M_2 \models^2 \alpha(\varphi)$. By the satisfaction condition again, but in the other direction, we obtain $M_1 \models \varphi$. □

**Corollary 3.8.** *By the second corridor of Prop.* 3.5*, we have that* **CPL** $\leq^{ER}$ **WPL**.

The first corridor of Prop. 3.5 only gives us a non-conservative morphism at the ER level.

### 3.1. Expressiveness, Sublogics and Paradoxes

The following concept is the model-theoretic analogue of Def. 2.14.

**Definition 3.9.** $\mathcal{R}_1 \leq^{SAT} \mathcal{R}_2$ *iff there is a model-expansive corridor* $(\alpha, \beta)\colon \mathcal{R}_1 \longrightarrow \mathcal{R}_2$.

$\mathcal{R}_1 \leq^{SAT} \mathcal{R}_2$ is read as "$\mathcal{R}_1$ is at most as expressive as $\mathcal{R}_2$" or "$\mathcal{R}_2$ is at least as expressive as $\mathcal{R}_1$". We also define $\equiv^{SAT}$ and $<^{SAT}$ in the same way as for ERs, and also the simple theoroidal variants. The following corollary expresses the conclusion of Prop. 3.7 with the new notions.

**Corollary 3.10.** *If* $\mathcal{R}_1 \leq^{SAT} \mathcal{R}_2$ *then* $\mathcal{S}_1 \leq^{ER} \mathcal{S}_2$*, where* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are ERs induced by corridors* $\mathcal{R}_1$ *and* $\mathcal{R}_2$*, respectively. Consequently,* $\mathcal{S}_1 \succeq \mathcal{S}_2$.

The corridor discussed in the previous section shows that **CPL** $\leq^{SAT}$ **WPL**, Béziau [4] considers the situation that a weaker logic is more expressive than a stronger one to be paradoxical.[12] Indeed, the situation

$$\begin{array}{ccc} \text{stronger logic} & \xleftarrow{\text{non-conservative inclusion}} & \text{weaker logic} \\ \text{less expressive} & \xrightarrow[\text{conservative encoding}]{} & \text{more expressive} \end{array}$$

seems to be quite natural and typical. The conservative ER morphism tells us that the weaker logic both has more consistency strength and is more expressive.

What does the non-conservative inclusion tell us? If its sentence translation is surjective, by Prop. 2.12, it reflects consistency. Altogether this means that both logics have equal consistency strength. However, there is no consequence in the context of expressiveness — expressiveness is defined in terms of *conservative* ER morphisms.

To sum up, Béziau considers the situation that a sublogic can be strictly more expressive than its superlogic to be paradoxical. Intuitively, one would expect a sublogic to be *less* expressive. However, we think that the problem is rather hidden in the concept of *sublogic*: in our view, a sublogic inclusion should be an injective *conservative* ER morphism. If we adopt this convention, then there is no paradox: the expressivity of any sublogic does not exceed the expressivity of its super logic. Concerning the example, the point is that the natural inclusion of **WPL** into **CPL** is not a sublogic morphism, because it is not conservative!

Note, however, there is a *simple theoroidal* model expansive corridor from **WPL** to **CPL**.

**Proposition 3.11.** **WPL** $\leq^{SAT\text{-}th}$ **CPL**.

*Proof.* The simple theoroidal corridor $(\alpha, \beta, \Delta)$ : **WPL** $\rightarrow$ **CPL** is defined as follows:

1. $\alpha$ maps injectively the sentences of **WPL** to propositional variables in **CPL**.
2. $\Delta$ axiomatises the propositional variables from the image of $\alpha$ according to the semantics of **WPL**, e.g.
   - $\alpha(\varphi \rightarrow \psi) \leftrightarrow (\alpha(\varphi) \rightarrow \alpha(\psi))$,
   - $\alpha(\varphi \wedge \psi) \leftrightarrow (\alpha(\varphi) \wedge \alpha(\psi))$,
   - $\alpha(\varphi \vee \psi) \leftrightarrow (\alpha(\varphi) \vee \alpha(\psi))$, and
   - $\alpha(\varphi) \rightarrow \neg\alpha(\neg\varphi)$.
3. For any **CPL**-model $M$ satisfying $\Delta$, $\beta(M)(\varphi)$ is defined as $M(\alpha(\varphi))$ for any sentence $\varphi$.

Note that the definition of $\beta$ gives precisely the satisfaction condition for this corridor. Moreover, the corridor is obviously model expansive. $\square$

Let us now come back to the relationship between intuitionistic and classical logic. Epstein [16] argues that there is no semantically faithful translation from classical to intuitionistic logic, at least for the standard semantics (and for his much

[12]The following considerations apply both to entailment relations and to rooms.

more specialised notion of semantic translation). Moreover, he asks whether there is a semantically faithful translation for other semantics. We answer this question positively: if we equip propositional logic with Boolean algebra semantics, there is a semantic translation:

**Theorem 3.12.** **CPL-BA** $\leq^{SAT}$ **IPL-HA**.

*Proof.* Assuming the same set $P$ of propositional symbols, recall that **CPL-BA**, **IPL-HA**, **CPL** and **IPL**, share the same sentences. We construct a model expansive corridor $(K, \beta)$ : **CPL-BA** $\rightarrow$ **IPL-HA** by extending Kolmogorov's translation $K$ of Ex. 2.4 with the following translation $\beta$ on the models.

For each valuation $M : P \rightarrow A$ of the propositional symbols into a Heyting algebra, $\beta(M) = r_A \circ M$ where $r_A : A \rightarrow R(A)$ is the canonical mapping to the Boolean algebra $R(A)$ of the regular elements of $A$. Recall that $a \in A$ is *regular* when $a = \neg\neg a$. From the theory of Heyting algebras [44, 7] we know that $R(A)$ is a Boolean algebra with

- $a \cap b = a \wedge b$,
- $a \cup b = \neg\neg(a \vee b)$,
- $a \rightarrow' b = a \rightarrow b$,
- $\neg' a = \neg a$,
- $\bot' = \bot$ and $\top' = \top$,

where $\cap, \cup, \rightarrow', \neg', \bot', \top'$ and $\wedge, \vee, \rightarrow, \neg, \bot, \top$, respectively, are the conjunctions, disjunctions, implications, negations, bottom, and top, in $R(A)$ and $A$, respectively.

Note that since $\beta(A) = A$ for any Boolean algebra $A$, we have that $\beta$ is surjective, hence our corridor will be model expansive.

Now all we have to show is the satisfaction condition for our corridor. Let $\varphi$ be any sentence and let $M : P \rightarrow A$ be any valuation of the propositional symbols into a Heyting algebra. We have to show that

$$M \models^{\mathbf{IPL-HA}} K(\varphi) \text{ if and only if } r_A \circ M \models^{\mathbf{CPL-BA}} \varphi$$

This is achieved by showing that

$$M(K(\varphi)) = (r_A \circ M)[\varphi]$$

where by $M(K(\varphi))$ we denote the evaluation of $K(\varphi)$ determined by $M$ in the Heyting algebra $A$ and by $(r_A \circ M)[\varphi]$ we denote the evaluation of $\varphi$ determined by $r_A \circ M$ in the Boolean algebra $R(A)$ of the regular elements of $A$. It is important to note here that in general $(r_A \circ M)[\varphi] \neq r_A(M(\varphi))$.

We prove that $M(K(\varphi)) = (r_A \circ M)[\varphi]$ by induction on the structure of $\varphi$ as follows.

- For the base case we consider any propositional symbol $p$. We have that $M(K(p)) = M(\neg\neg p) = \neg\neg M(p) = r_A(M(p)) = (r_A \circ M)[p]$.
- For the inductive step corresponding to conjunction, we assume the property holds for $\varphi_1$ and $\varphi_2$ and prove it for $\varphi_1 \wedge \varphi_2$. We have that

$$\begin{aligned} M(K(\varphi_1 \wedge \varphi_2)) &= M(\neg\neg(K(\varphi_1) \wedge K(\varphi_2))) \\ &= \neg\neg M(K(\varphi_1) \wedge K(\varphi_2)) \\ &= \neg\neg(M(K(\varphi_1)) \wedge M(K(\varphi_2))) \\ &= r_A(M(K(\varphi_1)) \wedge M(K(\varphi_2))). \end{aligned}$$

Because $r_A$ is homomorphism of Heyting algebras (i.e. function which preserves the interpretations of the conjunctions, disjunctions, implications, negations, bottom and top) [44, 7], we have that $r_A(M(K(\varphi_1)) \wedge M(K(\varphi_2))) = r_A(M(K(\varphi_1))) \cap r_A(M(K(\varphi_2)))$. By the induction hypothesis $M(K(\varphi_i)) = (r_A \circ M)[\varphi_i]$ are regular elements, hence $r_A(M(K(\varphi_i))) = M(K(\varphi_i))$ and thus

$$\begin{aligned} r_A(M(K(\varphi_1))) \cap r_A(M(K(\varphi_2))) &= (r_A \circ M)[\varphi_1] \cap (r_A \circ M)[\varphi_2] \\ &= (r_A \circ M)[\varphi_1 \wedge \varphi_2]. \end{aligned}$$

This means that $M(K(\varphi_1 \wedge \varphi_2)) = (r_A \circ M)[\varphi_1 \wedge \varphi_2]$.
- The induction steps corresponding to disjunction and implications get similar proofs as for conjunction, based upon the homomorphism property of $r_A$ — we omit the details here. We also omit the (rather trivial) induction steps corresponding to falsum and truth.
- For the induction step corresponding to negation, we assume the property holds for $\varphi$ and prove it for $\neg\varphi$. We have then:

$$\begin{aligned} M(K(\neg\varphi)) &= M(\neg K(\varphi)) = \neg M(K(\varphi)) \\ &= \neg(r_A \circ M)[\varphi] \\ &= \neg'(r_A \circ M)[\varphi] = (r_A \circ M)[\neg\varphi]. \end{aligned}$$

□

**Corollary 3.13. CPL $\leq^{ER}$ IPL.**

*Proof.* It is enough to show that Kolmogorov's translation is conservative as ER morphism. This follows because

1. $\Gamma \vdash^{\mathbf{CPL}} \varphi$ if and only if $\Gamma \models^{\mathbf{CPL-BA}} \varphi$, and
2. $\Gamma \vdash^{\mathbf{IPL}} \varphi$ if and only if $\Gamma \models^{\mathbf{IPL-HA}} \varphi$.

The second equivalence represents the completeness theorem for intuitionistic (propositional) logic with Heyting algebra semantics [44, 20]. The first equivalence is justified as follows.

Since the sentences and the proof calculus of **CPL** and **CPL-BA** coincide, the implication from the left to the right represents the soundness of the classical propositional proof calculus with respect to the **CPL-BA** semantics. This is exactly the soundness of the intuitionistic proof calculus (valid for any Heyting algebra) plus the soundness of the $\neg\neg$-elimination rule (valid only for Boolean algebras).

For the implication from the right to the left, assume that $\Gamma \models^{\mathbf{CPL-BA}} \varphi$. Since any valuation of the propositional variables into the Boolean algebra with two elements is also a model of **CPL-BA**, we obviously have that $\Gamma \models^{\mathbf{CPL}} \varphi$. By the completeness theorem of classical propositional calculus [44, 46] we have that $\Gamma \vdash^{\mathbf{CPL}} \varphi$. □

There is an obvious corridor **IPL-HA** $\rightarrow$ **CPL-BA**, which is the identity on sentences and construes each Boolean algebra as a Heyting algebra. However, the corridor is not model expansive, and the induced ER morphism is not conservative. An interesting open question is whether a model-expansive corridor **IPL-HA** $\rightarrow$ **CPL-BA** exists.

### 3.2. Logical Connectives

Logical connectives can also be defined at the semantic level. A room is said to *have a semantic connective* if it is possible to define an operation on sentences with the specified properties.

| connective | defining property |
|---|---|
| semantic disjunction $\vee$ | $M \models \varphi \vee \psi$ iff $M \models \varphi$ or $M \models \psi$ |
| semantic conjunction $\wedge$ | $M \models \varphi \wedge \psi$ iff $M \models \varphi$ and $M \models \psi$ |
| semantic implication $\rightarrow$ | $M \models \varphi \rightarrow \psi$ iff $M \models \varphi$ implies $M \models \psi$ |
| semantic negation $\neg$ | $M \models \neg\varphi$ iff $M \not\models \varphi$ |
| semantic truth $\top$ | $M \models \top$ |
| semantic falsum $\bot$ | $M \not\models \bot$ |

While **CPL** has all the semantic connectives as listed above (indeed, they coincide with the proof-theoretic ones), **IPL-HA** and **IPL-K** only have conjunction, truth and falsum.

Using the semantic connectives, the notion of *propositionally schematic corridor* can be introduced entirely parallel to the definition of propositionally schematic ER morphism in Def. 2.7.

**Definition 3.14.** *A logic room is* Henkin-sound*, if every satisfiable theory is consistent, and* Henkin-complete*, if the converse holds.*

We now have a further means for obtaining consistency:

**Proposition 3.15.** *A sound logic room that has a falsum which is both semantic and proof-theoretic is Henkin-sound. A Henkin-sound logic room with non-empty model class is consistent.*

*Proof.* Let $\Gamma$ be satisfiable, thus having a model $M$. Since $M \not\models \bot$, $\Gamma \not\models \bot$. By soundness, $\Gamma \nvdash \bot$. Hence, $\Gamma$ is consistent. The second result follows using the empty theory. $\square$

**Lemma 3.16.** *In presence of proof-theoretic negation, falsum and truth,*

$$\neg\bot \dashv\vdash \top \text{ and } \neg\top \dashv\vdash \bot.$$

Henkin-style completeness proofs rely on the following fact:

**Proposition 3.17.** *If a Henkin-complete room has negation, truth and falsum which are both proof-theoretic and semantic then it is complete.*

*Proof.* Let $\Gamma \models \varphi$. Then $\Gamma \cup \{\neg\varphi\}$ is not satisfiable, hence inconsistent. From $\Gamma \cup \{\neg\varphi\} \vdash \bot$, we obtain $\Gamma \cup \{\neg\varphi\} \vdash \neg\neg\bot$ and hence $\Gamma \cup \{\neg\bot\} \vdash \varphi$. By Lemma 3.16, $\Gamma \cup \{\top\} \vdash \varphi$. By the properties of $\top$ and transitivity, we get $\Gamma \vdash \varphi$. $\square$

**Proposition 3.18.** *Consider a corridor* $(\alpha, \beta) : (S_1, \mathcal{M}_1, \models^1) \to (S_2, \mathcal{M}_2, \models^2)$.

1. *If the image of* $\alpha$ *is closed under semantic connectives and the corridor is model-expansive then* $(\alpha, \beta)$ *reflects semantic connectives.*
2. *If* $\alpha$ *is surjective then* $(\alpha, \beta)$ *transports semantic connectives.*

*Proof.* 1. We do this proof only for conjunction, for the other connectives the corresponding proofs are similar. Consider $\varphi, \varphi' \in S_1$. Because the image of $\alpha$ is closed under conjunction, there exists $\psi \in S_1$ such that $\alpha(\psi) = \alpha(\varphi) \wedge \alpha(\varphi')$. We have to prove that for any model $M_1 \in \mathcal{M}_1$,

$$M_1 \models^1 \psi \ \text{ iff } \ (M_1 \models^1 \varphi \ \text{ and } \ M_1 \models^1 \varphi')$$

Because the corridor is model-expansive there exists $M_2 \in \mathcal{M}_2$ such that $M_1 = \beta(M_2)$. By the satisfaction condition we have:

- $M_1 \models^1 \psi$ if and only if $M_2 \models^2 \alpha(\psi)$,
- $M_1 \models^1 \varphi$ if and only if $M_2 \models^2 \alpha(\varphi)$, and
- $M_1 \models^1 \varphi'$ if and only if $M_2 \models^2 \alpha(\varphi')$.

The conclusion follows from the above observations and from the fact that $\alpha(\psi) = \alpha(\varphi) \wedge \alpha(\varphi')$ which implies that $M_2 \models^2 \alpha(\psi)$ iff $M_2 \models^2 \alpha(\varphi)$ and $M_2 \models^2 \alpha(\varphi')$.

2. Again, we do this proof only for conjunction. Consider $\varphi_2, \varphi_2' \in S_2$. Since $\alpha$ is surjective there are $\varphi_1, \varphi_1' \in S_1$ such that $\varphi_2 = \alpha(\varphi_1)$ and $\varphi_2' = \alpha(\varphi_1')$. We show that $\alpha(\varphi_1 \wedge \varphi_1')$ is the semantic conjunction of $\varphi_2$ and $\varphi_2'$. For this we have to show that for any model $M_2 \in \mathcal{M}_2$ we have

$$M_2 \models^2 \alpha(\varphi_1 \wedge \varphi_1') \ \text{ iff } \ (M_2 \models^2 \varphi_2 \ \text{ and } \ M_2 \models^2 \varphi_2')$$

which by the satisfaction condition is equivalent to

$$\beta(M_2) \models^1 \varphi_1 \wedge \varphi_1' \ \text{ iff } \ (\beta(M_2) \models^1 \varphi_1 \ \text{ and } \ \beta(M_2) \models^1 \varphi_1')$$

But this holds by the definition of the semantic conjunction. $\square$

A notion of equivalence for rooms is given in [40] — however, here we consider its weaker version.

**Definition 3.19.** *Two rooms* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *are* weakly equivalent*, if there are morphisms* $(\alpha_1, \beta_1) \colon \mathcal{R}_1 \longrightarrow \mathcal{R}_2$ *and* $(\alpha_2, \beta_2) \colon \mathcal{R}_2 \longrightarrow \mathcal{R}_1$ *such that* $\varphi \models\mid \alpha_2(\alpha_1(\varphi))$. *(*$\models\mid$ *means semantic entailment in both directions.)*

**Proposition 3.20.** *Weakly equivalent rooms have the same semantic connectives.*

*Proof.* This follows by a slightly more general version of the second part of Prop. 3.18 which assumes the surjectivity of $\alpha$ modulo the semantical equivalence $\models\mid$, rather than strict surjectivity. It is easy to note that the proof of this slight generalisation remains essentially the same as the proof of the second part of Prop. 3.18. Hence we have that both corridors $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ transport semantic connectives, which means that the rooms have the same semantic connectives. $\square$

## 4. Institutions and Logics

When carefully inspecting the development that we have made so far, one will recognise that in all the examples, the set of propositional symbols remains unspecified, but it needs to be a set that is fixed once and for all. If this set is finite, we are rather inflexible, since the number of propositional variables is bounded. This may also lead to severe technical difficulties, for example results such as Prop. 3.11 may fail. If it is infinite, both models and axiom sets $\Delta$ used for simple theoroidal morphisms typically have to be infinite as well, which is unsatisfactory. Hence, it makes sense to let the alphabet of symbols *vary*. What happens then when we *change* the set of propositional symbols? For example, [2, 3] mention the property of an entailment relation to be stable under mappings of the propositional variables. The concepts of *signature* (i.e. context of non-logical vocabulary) and *signature morphism* allow for generalisation of this property to an abstract logic, without commitment to any particular nature of the signatures. We hence arrive at a *category* of signatures and signature morphisms, which indexes entailment relations and rooms. That is, there are ER morphisms and corridors not only for the translation between logics, but also for the translation along signature morphisms within one logic.[13]

**Definition 4.1.** *An* entailment system (or $\Pi$-institution) [38, 18] *is a functor* $\mathcal{E}\colon \mathbb{S}ign^{\mathcal{E}} \longrightarrow \mathbb{ER}$. *This amounts to:*

- *a category* $\mathbb{S}ign^{\mathcal{E}}$ *of signatures and signature morphisms,*
- *for each signature* $\Sigma \in |\mathbb{S}ign^{\mathcal{E}}|$*, an entailment relation* $(\mathsf{Sen}(\Sigma), \vdash^{\mathcal{E}}_{\Sigma})$*,*
- *for each signature morphism* $\sigma\colon \Sigma_1 \longrightarrow \Sigma_2$*, a entailment relation morphism from* $(\mathsf{Sen}^{\mathcal{E}}(\Sigma_1), \vdash^{\mathcal{E}}_{\Sigma_1})$ *to* $(\mathsf{Sen}^{\mathcal{E}}(\Sigma_2), \vdash^{\mathcal{E}}_{\Sigma_2})$*, that is, a sentence translation map* $\mathsf{Sen}^{\mathcal{E}}(\Sigma_1) \to \mathsf{Sen}^{\mathcal{E}}(\Sigma_2)$ *preserving* $\vdash^{\mathcal{E}}$*. By abuse of notation, we will also denote this map by* $\sigma$*.*

**Definition 4.2.** *An* institution[14] [28] *is a functor* $\mathcal{I}\colon \mathbb{S}ign \longrightarrow \mathbb{R}oom$*. This amounts to:*

- *a category* $\mathbb{S}ign^{\mathcal{I}}$ *of signatures and signature morphisms,*
- *for each signature* $\Sigma \in |\mathbb{S}ign^{\mathcal{I}}|$*, a room* $(\mathsf{Sen}^{\mathcal{I}}(\Sigma), \mathsf{Mod}^{\mathcal{I}}(\Sigma), \models^{\mathcal{I}}_{\Sigma})$*,*
- *for each signature morphism* $\sigma\colon \Sigma_1 \longrightarrow \Sigma_2$*, a corridor*

$$(\mathsf{Sen}^{\mathcal{I}}(\Sigma_1), \mathsf{Mod}^{\mathcal{I}}(\Sigma_1), \models^{\mathcal{I}}_{\Sigma_1}) \to (\mathsf{Sen}^{\mathcal{I}}(\Sigma_2), \mathsf{Mod}^{\mathcal{I}}(\Sigma_2), \models^{\mathcal{I}}_{\Sigma_2}).$$

  *This amounts to:*
  1. *a sentence translation map* $\sigma\colon \mathsf{Sen}^{\mathcal{I}}(\Sigma_1) \longrightarrow \mathsf{Sen}^{\mathcal{I}}(\Sigma_2)$*, and*
  2. *a model reduction map* $\mathsf{Mod}^{\mathcal{I}}(\sigma)\colon \mathsf{Mod}^{\mathcal{I}}(\Sigma_2) \longrightarrow \mathsf{Mod}^{\mathcal{I}}(\Sigma_1)$ *(sometimes denoted* $\_\restriction_{\sigma}$*).*

---

[13] For this section, we assume the reader is familiar with basic notions from category theory; for instance, see [1, 35] for introductions to this subject. By way of notation, $|\mathbb{C}|$ denotes the class of objects of a category $\mathbb{C}$, and composition is denoted by "$\circ$".

[14] Hence, an institution consists of rooms and corridors, like in real life.

*such that the satisfaction condition holds:*

$$M' \models^{\mathcal{I}}_{\Sigma'} \mathsf{Sen}^{\mathcal{I}}(\sigma)(\varphi) \quad \textit{if and only if} \quad \mathsf{Mod}^{\mathcal{I}}(\sigma)(M') \models^{\mathcal{I}}_{\Sigma} \varphi$$

*for each* $M' \in |\mathsf{Mod}^{\mathcal{I}}(\Sigma_2)|$ *and* $\varphi \in \mathsf{Sen}^{\mathcal{I}}(\Sigma_1)$*.*

*When* $M_1 = M_2\restriction_\sigma$ *we say that* $M_1$ *is a* $\sigma$-reduct *of* $M_2$ *or that* $M_2$ *is a* $\sigma$-expansion *of* $M_1$*.*

*A* logic *is an institution together with an entailment system agreeing on their signature and sentence parts. Usually, a logic is required to be* sound*, that is,* $\Gamma \vdash_\Sigma \varphi$ *implies* $\Gamma \models_\Sigma \varphi$*. If also the converse holds, the logic is complete. Note that this coincides with the corresponding properties of the logic rooms for all the signatures.*

We can now define entailment systems and institutions for **CPL**, **CPL-BA**, **IPL-HA**, **IPL-K**, **WPL** and so on. The category of signatures is just the category of sets and functions. For each signature $\Sigma$, the corresponding ER (respectively: room) is built using propositional variables from the set $\Sigma$. This ensures for instance that models in **CPL** are finite for finite signatures. For a signature morphism $\sigma\colon \Sigma_1 \longrightarrow \Sigma_2$, sentence translation replaces propositional variables in the sentences according to $\sigma$. A $\Sigma_2$-model, which is usually a map $M_2$ from $\Sigma_2$ into some semantic domain $\mathcal{D}$, is reduced to $M_2\restriction_\sigma = M_2 \circ \sigma$.

### 4.1. Comorphisms

Relationships between institutions (and entailment systems) are captured mathematically by 'institution morphisms', of which there are several variants, each yielding a category under a canonical composition. For the purposes of this paper, institution comorphisms [29] seem technically most convenient, since they capture the intuition of coding of one logical system into another one. (The original notion from [28] works well for 'forgetful' morphisms from one institution to another having less structure.)

**Definition 4.3.** *Given entailment systems* $\mathcal{E}$ *and* $\mathcal{F}$*, an* entailment system comorphism $(\Phi, \alpha)\colon \mathcal{E} \longrightarrow \mathcal{F}$ *consists of*

- *a functor* $\Phi\colon \mathbb{S}ign^{\mathcal{E}} \longrightarrow \mathbb{S}ign^{\mathcal{F}}$*, and*
- *for each* $\Sigma \in |\mathbb{S}ign^{\mathcal{E}}|$*, an ER morphism*

$$\alpha_\Sigma\colon (\mathsf{Sen}^{\mathcal{E}}(\Sigma), \vdash^{\mathcal{E}}_{\Sigma}) \longrightarrow (\mathsf{Sen}^{\mathcal{F}}(\Phi(\Sigma)), \vdash^{\mathcal{F}}_{\Phi(\Sigma)})$$

*that is natural in* $\Sigma$*.*

*Given institutions* $\mathcal{I}$ *and* $\mathcal{J}$*, an* institution comorphism $(\Phi, \alpha, \beta)\colon \mathcal{I} \longrightarrow \mathcal{J}$ *consists of*

- *a functor* $\Phi\colon \mathbb{S}ign^{\mathcal{I}} \longrightarrow \mathbb{S}ign^{\mathcal{J}}$*, and*
- *for each* $\Sigma \in |\mathbb{S}ign^{\mathcal{I}}|$*, a corridor*

$$(\alpha_\Sigma, \beta_\Sigma) : (\mathsf{Sen}^{\mathcal{I}}(\Sigma), \mathsf{Mod}^{\mathcal{I}}(\Sigma), \models^{\mathcal{I}}_{\Sigma}) \to (\mathsf{Sen}^{\mathcal{J}}(\Phi(\Sigma)), \mathsf{Mod}^{\mathcal{J}}(\Phi(\Sigma)), \models^{\mathcal{J}}_{\Phi(\Sigma)})$$

*that is natural in* $\Sigma$*.*

*With the natural compositions and identities, this gives categories* $\mathbb{C}oCons$ *and* $\mathbb{C}oIns$*, respectively. Logic comorphisms are institution comorphisms with signature and sentence translations that yield comorphisms between the entailment systems of the logics. Simple theoroidal variants of these notions are defined in the obvious way.*

**Example 4.4.** The standard translation from propositional modal logic to first-order logic is an institution comorphism. In signatures, each propositional symbol is translated to a unary predicate, and a binary predicate (the accessibility relation) is added to all signatures. Modalities are expressed using quantification. A first-order model (of a translated signature) can be constructed as a Kripke structure. This translation can be extended to first-order modal logic.

Practically all of the notions that we have introduced so far, like consistency, conservativeness, the model-expansion property, expressiveness, sublogic, equivalences of logics, proof-theoretic and model-theoretic connectives, easily lift from the non-indexed to the indexed level, with the following provisos: A *theory* is a pair $(\Sigma, \Gamma)$ where $\Gamma$ is a set of $\Sigma$-sentences. An entailment system (respectively: institution) is consistent, if each signature has a consistent theory. A sublogic morphism needs to have a signature translation $\Phi$ that is an embedding of categories, that is, injective on objects, full and faithful. For a logic equivalence, $\Phi$ needs to be an equivalence of categories.

### 4.2. Quantification

Quantification is an important concept that can be handled much better at the indexed level. Lawvere [36, 37] defined quantification as adjoint to substitution. Here we define quantification as adjoint to sentence translation along a class $\mathcal{D}$ of signature morphisms, which typically introduce new constants to serve as quantification "variables":

**Definition 4.5.** *An entailment system* has proof-theoretic universal (respectively: existential) $\mathcal{D}$-quantification *for a class* $\mathcal{D}$ *of signature morphisms, if for all signature morphisms* $\sigma\colon \Sigma \longrightarrow \Sigma' \in \mathcal{D}$*, there is a function* $(\forall\sigma)_{\_}$ *(respectively:* $(\exists\sigma)_{\_}$*) from* $\mathsf{Sen}(\Sigma')$ *to* $\mathsf{Sen}(\Sigma)$ *such that*

$$\Gamma \vdash_{\Sigma} (\forall\sigma)\varphi' \textit{ iff } \sigma(\Gamma) \vdash_{\Sigma'} \varphi'$$

*and respectively:*

$$(\exists\sigma)\Gamma' \vdash_{\Sigma} \varphi \textit{ iff } \Gamma' \vdash_{\Sigma} \sigma(\varphi)$$

**Definition 4.6.** *An institution has* semantic universal (existential) $\mathcal{D}$-quantification [47] *for a class* $\mathcal{D}$ *of signature morphisms, if for all signature morphisms* $\sigma\colon \Sigma \longrightarrow \Sigma' \in \mathcal{D}$ *and each* $\Sigma'$*-sentence* $\varphi$*, there is a* $\Sigma$*-sentence* $\forall\sigma.\varphi$ *(respectively:* $\exists\sigma.\varphi$*) such that* $M \models_{\Sigma} \forall\sigma.\varphi$ *iff* $M' \models \varphi$ *for all* $\sigma$*-expansions* $M'$ *of* $M$ *(respectively:* $M \models_{\Sigma} \exists\sigma.\varphi$ *iff* $M' \models \varphi$ *for some* $\sigma$*-expansion* $M'$ *of* $M$*).*

These definitions accommodate quantification over any entities which are named in the relevant concept of signature. For conventional model theory, this includes second-order quantification by taking $\mathcal{D}$ to be all extensions of signatures by operation and relation symbols. First-order quantification is modelled by taking $\mathcal{D}$ to be all extensions of signatures by finite sets of constants, or more abstractly, by taking $\mathcal{D}$ to be all the *representable signature morphisms* [13, 14], building on the observation that an assignment for a set of (first order) variables corresponds to a model morphism from the free (term) model over that set of variables.

**Example 4.7.** Let **FOL** be the usual institution of single-sorted classical first-order logic, and **MSFOL** its many-sorted variant [28]. Then **FOL** and **MSFOL** have both proof-theoretic and semantic universal and existential first-order quantification (i.e., $\mathcal{D}$-quantification with $\mathcal{D}$ as defined above for first-order quantification).

For (single-sorted) intuitionistic first-order logic, the Heyting algebra variant is denoted by **IFOL-CHA**. The interpretation of universal and existential quantifiers needs possibly infinite meets and joins, hence a complete Heyting algebra is required. Kripke semantics comes in two variants: with constant domains (**IFOL-CD**) and possibly increasing domains (**IFOL-HQ**) [20]. All these three logics have proof-theoretic universal and existential first-order quantification — actually, the conditions in Def. 4.5 are very close to the usual quantifier rules. Moreover, all three lack semantic existential (first-order) quantification: consider a constant-domain Kripke model with two worlds $w_1$ and $w_2$, two individuals $a_1$ and $a_2$, and a unary predicate $P$ such that in world $w_i$, $P(a_i)$ holds, but $P(a_{3-i})$ does not hold. In this model, $\exists x\,.\,P(x)$ holds, but the model does not have an expansion interpreting a constant $c$ such that $P(c)$ holds: $P(c)$ would have to hold in all worlds. This provides a counterexample for both **IFOL-CD** and **IFOL-HQ**, and it can be turned into one for **IFOL-CHA** by using the Heyting (even Boolean) algebra $\mathcal{P}(\{w_1, w_2\})$ and letting $P(a_i)$ have the truth value $\{w_i\}$. Actually, this counterexample is also a counterexample for **FOL-BA**, first-order logic with Boolean-valued models.

Finally, the three institutions have semantic universal first-order quantification. For **IFOL-CHA**, this follows since the meet of a set is $\top$ iff all its members are $\top$. For **IFOL-CD**, this follows since universal quantification over worlds and over domain elements can be interchanged. This carries over to **IFOL-HQ** for quantification w.r.t. all extensions of signatures with flexible constants. However, for extensions with rigid constants, semantic universal quantification cannot be given in general: consider a Kripke model with two worlds $w$ and $v$, with $w \leq v$, one individual $a$ with $P(a)$ for world $w$ and two individuals $a, b$ with $P(a)$ but not $P(b)$ for world $v$. In this model, $\forall x\,.\,P(x)$ does not hold, but any expansion interpreting a rigid constant $c$ must interpret $c$ by $a$, hence $P(c)$ holds in it. □

One can define a notion of first-order schematic comorphism much in the same way as the notion of propositionally schematic morphisms in Sect. 2.1, if one fixes a canonical choice for the class $\mathcal{D}$ of signature morphisms (e.g. the representable signature morphisms).

We now study the interaction of quantification and comorphisms.

**Proposition 4.8.** *Let* $(\Phi, \alpha) : (\mathbb{S}ign_1, \mathsf{Sen}_1, \vdash^1) \to (\mathbb{S}ign_2, \mathsf{Sen}_2, \vdash^2)$ *be a conservative entailment system comorphism and let* $\mathcal{D}_1 \subseteq \mathbb{S}ign_1$ *and* $\mathcal{D}_2 \subseteq \mathbb{S}ign_2$ *be classes of signature morphisms such that* $\Phi(\mathcal{D}_1) \subseteq \mathcal{D}_2$*.*

1. *The comorphism* $(\Phi, \alpha)$ *reflects proof-theoretic* $\mathcal{D}_2$*-quantification to proof-theoretic* $\mathcal{D}_1$*-quantification if the image of* $\alpha$ *is closed under the former.*
2. *The comorphism* $(\Phi, \alpha)$ *transports proof-theoretic* $\mathcal{D}_1$*-quantification to proof-theoretic* $\mathcal{D}_2$*-quantification if* $\Phi(\mathcal{D}_1) = \mathcal{D}_2$*, and* $\alpha_\Sigma$*'s are surjective.*

*Proof.* 1. We carry out the proof only for universal quantification, the proof for existential quantification is similar. Consider $(\sigma : \Sigma \to \Sigma') \in \mathcal{D}_1$ and $\varphi' \in \mathsf{Sen}_1(\Sigma')$. Because the image of $\alpha$ is closed under proof-theoretic $\mathcal{D}_2$-quantification there exists $\psi \in \mathsf{Sen}_1(\Sigma)$ such that $\alpha_\Sigma(\psi) = (\forall\Phi(\sigma))\alpha_{\Sigma'}(\varphi')$. We show that $\psi$ defines $(\forall\sigma)\varphi'$, which amounts to showing that

$$\Gamma \vdash^1_\Sigma \psi \ \text{ iff } \ \sigma(\Gamma) \vdash^1_{\Sigma'} \varphi'$$

Because the comorphism is conservative we have:

- $\Gamma \vdash^1_\Sigma \psi$ is equivalent to $\alpha_\Sigma(\Gamma) \vdash^2_{\Phi(\Sigma)} \alpha_\Sigma(\psi) = (\forall\Phi(\sigma))\alpha_{\Sigma'}(\varphi')$.
- $\sigma(\Gamma) \vdash^1_{\Sigma'} \varphi'$ is equivalent to $\alpha_{\Sigma'}(\sigma(\Gamma)) \vdash^2_{\Phi(\Sigma')} \alpha_{\Sigma'}(\varphi')$. Note that by the naturality of $\alpha$ we have $\alpha_{\Sigma'}(\sigma(\Gamma)) = \Phi(\sigma)(\alpha_\Sigma(\Gamma))$.

The conclusion now follows by the quantification property

$$\alpha_\Sigma(\Gamma) \vdash^2_{\Phi(\Sigma)} (\forall\Phi(\sigma))\alpha_{\Sigma'}(\varphi') \ \text{ iff } \ \alpha_{\Sigma'}(\sigma(\Gamma)) \vdash^2_{\Phi(\Sigma')} \Phi(\sigma)(\alpha_\Sigma(\Gamma))$$

2. Again, we carry out the proof only for universal quantification. Consider $(\sigma_2 : \Sigma_2 \to \Sigma_2') \in \mathcal{D}_2$ and $\varphi_2' \in \mathsf{Sen}_2(\Sigma_2')$. Because $\Phi(\mathcal{D}_1) = \mathcal{D}_2$ there exists $(\sigma_1 : \Sigma_1 \to \Sigma_1') \in \mathcal{D}_1$ such that $\Phi(\sigma_1) = \sigma_2$ (which implies $\Phi(\Sigma_1) = \Sigma_2$ and $\Phi(\Sigma_1') = \Sigma_2'$). Because $\alpha_{\Sigma_1'}$ is surjective there exists $\varphi_1' \in \mathsf{Sen}_1(\Sigma_1')$ such that $\varphi_2' = \alpha_{\Sigma_1'}(\varphi_1')$. We show that $\alpha_{\Sigma_1}((\forall\sigma_1)\varphi_1')$ defines $(\forall\sigma_2)\varphi_2'$, which amounts to showing that

$$\Gamma_2 \vdash^2_{\Sigma_2} \alpha_{\Sigma_1}((\forall\sigma_1)\varphi_1') \ \text{ iff } \ \sigma_2(\Gamma_2) \vdash^2_{\Sigma_2'} \varphi_2'$$

Because $\alpha_{\Sigma_1}$ is surjective there exists $\Gamma_1$ such that $\alpha_{\Sigma_1}(\Gamma_1) = \Gamma_2$. By the naturality of $\alpha$ we have that $\sigma_2(\Gamma_2) = \sigma_2(\alpha_{\Sigma_1}(\Gamma_1)) = \alpha_{\Sigma_1'}(\sigma_1(\Gamma_1))$. Recall also that $\varphi_2' = \alpha_{\Sigma_1'}(\varphi_1')$. Hence, since the comorphism is conservative, we have:

- $\Gamma_2 \vdash^2_{\Sigma_2} \alpha_{\Sigma_1}((\forall\sigma_1)\varphi_1')$ is equivalent to $\Gamma_1 \vdash^1_{\Sigma_1} (\forall\sigma_1)\varphi_1'$, and
- $\sigma_2(\Gamma_2) \vdash^2_{\Sigma_2'} \varphi_2'$ is equivalent to $\sigma_1(\Gamma_1) \vdash^1_{\Sigma_1'} \varphi_1'$.

The conclusion now follows by the quantification property

$$\Gamma_1 \vdash^1_{\Sigma_1} (\forall\sigma_1)\varphi_1' \ \text{ iff } \ \sigma_1(\Gamma_1) \vdash^1_{\Sigma_1'} \varphi_1'$$

$\square$

**Definition 4.9.** *Let* $(\Phi, \alpha, \beta) : \mathcal{I} \longrightarrow \mathcal{J}$ *be an institution comorphism. Then* $(\Phi, \alpha, \beta)$ *is said to be* weakly exact*, if for each signature morphism* $\sigma : \Sigma \longrightarrow \Sigma'$ *in* $\mathcal{I}$*, and for* *any two models* $M'_1 \in \mathsf{Mod}^{\mathcal{I}}(\Sigma')$ *and* $M_2 \in \mathsf{Mod}^{\mathcal{J}}(\Phi(\Sigma_1))$ *with* $M'_1{\upharpoonright}_\sigma = \beta_\Sigma(M_2)$, *there is a* $M'_2 \in \mathsf{Mod}^{\mathcal{J}}(\Phi(\Sigma'))$ *with* $\beta_{\Sigma'}(M'_2) = M'_1$ *and* $M'_2{\upharpoonright}_{\Phi(\sigma)} = M_2$.

**Proposition 4.10.** *Let* $(\Phi, \alpha, \beta) : (\mathbb{S}ign_1, \mathsf{Sen}_1, \mathsf{Mod}_1, \models^1) \to (\mathbb{S}ign_2, \mathsf{Sen}_2, \mathsf{Mod}_2, \models^2)$ *be a weakly exact institution comorphism and let* $\mathcal{D}_1 \subseteq \mathbb{S}ign_1$ *and* $\mathcal{D}_2 \subseteq \mathbb{S}ign_2$ *be classes of signature morphisms such that* $\Phi(\mathcal{D}_1) \subseteq \mathcal{D}_2$.

1. *The comorphism* $(\Phi, \alpha, \beta)$ *reflects semantic* $\mathcal{D}_2$*-quantification to semantic* $\mathcal{D}_1$*-quantification if it is model-expansive and the image of* $\alpha$ *is closed under the former.*
2. *The comorphism* $(\Phi, \alpha, \beta)$ *transports semantic* $\mathcal{D}_1$*-quantification to semantic* $\mathcal{D}_2$*-quantification if* $\Phi(\mathcal{D}_1) = \mathcal{D}_2$ *and* $\alpha_\Sigma$*'s are surjective.*

*Proof.* Let us consider only universal quantification since the proofs for existential quantification are similar.

1. The construction of the reflected quantification is the same as the corresponding construction in the first part of the proof of Prop. 4.8, we also use here the same notations. We thus have to show that for all $\Sigma$-models $M$

$$M \models^1_\Sigma \psi \;\text{ iff }\; (M' \models^1_{\Sigma'} \varphi' \text{ for all } \sigma\text{-expansions } M' \text{ of } M)$$

Because the comorphism is model-expansive, there exists $M_2$ such that $M = \beta_\Sigma(M_2)$. By the satisfaction condition we have that

$$M \models^1_\Sigma \psi \;\text{ iff }\; M_2 \models^2_{\Phi(\Sigma)} \forall\Phi(\sigma).\alpha_{\Sigma'}(\varphi')$$

But

$$M_2 \models^2_{\Phi(\Sigma)} \forall\Phi(\sigma).\alpha_{\Sigma'}(\varphi') \;\text{ iff }\; (M'_2 \models^2_{\Phi(\Sigma')} \alpha_{\Sigma'}(\varphi') \text{ for all } \Phi(\sigma)\text{-expansions } M'_2 \text{ of } M_2)$$

By the satisfaction condition for the comorphism the latter equivalence means

$$M_2 \models^2_{\Phi(\Sigma)} \forall\Phi(\sigma).\alpha_{\Sigma'}(\varphi') \;\text{ iff }\; (\beta_{\Sigma'}(M'_2) \models^1_{\Sigma'} \varphi' \text{ for all } \Phi(\sigma)\text{-expansions } M'_2 \text{ of } M_2)$$

The conclusion follows by using the weakly exactness property since any $\sigma$-expansion $M'$ determines a $\Phi(\sigma)$-expansion $M'_2$, and conversely, each $\Phi(\sigma)$-expansion $M'_2$ determines a $\sigma$-expansion $M'$, such that $\beta_{\Sigma'}(M'_2) = M'$.

2. The construction of the transported quantification is the same as the corresponding construction in the second part of the proof of Prop. 4.8, we also use here the same notations. We thus have to show that

$$M_2 \models^2_{\Sigma_2} \alpha_{\Sigma_1}(\forall\sigma_1.\varphi'_1) \;\text{ iff }\; (M'_2 \models^2_{\Sigma'_2} \varphi'_2 \text{ for all } \sigma_2\text{-expansions } M'_2 \text{ of } M_2)$$

First let us note that by the satisfaction condition we have:

- $M_2 \models^2_{\Sigma_2} \alpha_{\Sigma_1}(\forall\sigma_1.\varphi'_1)$ iff $\beta_{\Sigma_1}(M_2) \models^1_{\Sigma_1} \forall\sigma_1.\varphi'_1$, and
- $M'_2 \models^2_{\Sigma'_2} \varphi'_2$ iff $\beta_{\Sigma'_1}(M'_2) \models^1_{\Sigma'_1} \varphi'_1$.

Now, by these equivalences, the implication from the left to the right follows from the fact that for all $M'_2$ such that $M'_2{\upharpoonright}_{\sigma_2} = M_2$, by the naturality of $\beta$ we have that $\beta_{\Sigma'_1}(M'_2){\upharpoonright}_{\sigma_1} = \beta_{\Sigma_1}(M_2)$.

For the implication from the right to the left, let $M'_1$ be any $\sigma_1$-expansion of $\beta_{\Sigma_1}(M_2)$. We have to show that $M'_1 \models^1_{\Sigma'_1} \varphi'_1$. Since the institution comorphism considered is weakly exact, there exists a $\Sigma'_2$-model $M'_2$ such that $\beta_{\Sigma'_1}(M'_2) = M'_1$ and $M'_2{\restriction}_{\sigma_2} = M_2$. Thus $M'_1 \models^1_{\Sigma'_1} \varphi'_1$ holds because $M'_2 \models^2_{\Sigma'_2} \varphi'_2$ and by the satisfaction condition for the comorphism. $\square$

## 5. Conclusions

Using morphisms between entailment relations and systems, and between rooms and institutions, we have shed some light on the notions of expressiveness, consistency strength and sublogic. It turns out that sublogics are less expressive than their superlogics only if a sufficiently strong notion of a sublogic is used. Consistency strength is a notion that behaves contravariantly w.r.t. expressiveness, that is, more expressive logics have less consistency strength. This is because weaker axiomatisations are less constraining and lead to more discriminatory strength.

We also have shown that logical structure such as connectives and quantifiers can be transported and reflected along such morphisms. Indeed, this kind of "borrowing" [11] of logical structure along morphisms has been studied in many different contexts. It is possible to borrow logical structure like proof theory, model theory, connectives and quantifiers, and also results like Craig interpolation, Beth definability, ultraproducts or the property of being a Łoś-institution, see [15] for details. Many of these concepts and results can only be made formal at the level of entailment systems and institutions, that is, when logics are considered with explicit indexing of their components (sets of sentences, classes of models, entailment, satisfaction) by signatures.

An interesting open question is the formalisation of "structurality" of translations between logics, such that translations flattening out the structure (like that in Prop. 2.25) are ruled out. However, the notions studied in the literature so far [48, 9] are clearly too limited here, as they focus on propositional connectives only. A proper notion would have to take into account also binding structures like quantification. The resulting notion of expressiveness then would define *logical frameworks* as used in the theorem proving community: they are logics with maximal expressiveness.

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