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NAMED MODELS IN COALGEBRAIC HYBRID LOGIC

LUTZ SCHRÖDER [1] AND DIRK PATTINSON [2]

[1] DFKI Bremen and Department of Computer Science, Universität Bremen
E-mail address: Lutz.Schroeder@dfki.de

[2] Department of Computing, Imperial College London
E-mail address: dirk@doc.ic.ac.uk

ABSTRACT. Hybrid logic extends modal logic with support for reasoning about individual states, designated by so-called nominals. We study hybrid logic in the broad context of coalgebraic semantics, where Kripke frames are replaced with coalgebras for a given functor, thus covering a wide range of reasoning principles including, e.g., probabilistic, graded, default, or coalitional operators. Specifically, we establish generic criteria for a given coalgebraic hybrid logic to admit named canonical models, with ensuing completeness proofs for pure extensions on the one hand, and for an extended hybrid language with local binding on the other. We instantiate our framework with a number of examples. Notably, we prove completeness of graded hybrid logic with local binding.

Introduction

Modal logics have traditionally played a central role in Computer Science, appearing, e.g., in the guise of temporal logics, program logics such as PDL, epistemic logics, and later as description logics. The development of modal logics has seen extensions along (at least) two axes: the enhancement of the expressive power of basic (relational) modal logic on the one hand, and the continual extension, beyond the purely relational realm, of the class of structures described using modal logics on the other hand. Hybrid logic falls into the first category, extending modal logic with the ability to reason about individual states in models. This feature, originally suggested by Prior and first studied in the context of tense logics and PDL (see [5] for references), is of particular relevance in knowledge representation languages and as such has found its way into modern description logics, where it is denoted by the letter $\mathcal{O}$ in the standard naming scheme [2].

Extensions along the second axis – semantics beyond Kripke structures and neighbourhood models – include various probabilistic modal logics, interpreted over probabilistic transition systems, graded modal logic over multigraphs [8], conditional logics over selection function frames [6], and coalition logic [17], interpreted over so-called game frames. As a unifying semantic bracket covering all these logics and many further ones, coalgebraic modal logic has emerged ([7] gives a survey). The scope of coalgebraic modal logic has recently been expanded to encompass nominals; we refer to the arising class of logics as *coalgebraic hybrid logics*. Existing results include a finite

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model result, an internalized tableaux calculus, and generic $PSPACE$ upper bounds, but are so far limited to logics that exclude frame conditions and local binding [14]. What is missing from this picture technically is a theory of *named canonical models* [5]. Named canonical models yield not only strong completeness of the basic hybrid logic, but also completeness of *pure extensions*, defined by axioms that do not contain propositional variables (but may contain nominals; e.g. in Kripke semantics, the pure axiom $\Diamond\Diamond i \to \Diamond i$, with i a *nominal*, defines transitive frames). Moreover, named canonical models establish completeness for an extended hybrid language with a local binding operator $\downarrow x.\, \phi(x)$, read as "the current state x satisfies $\phi(x)$". Both pure extensions and the language with $\downarrow$ (not addressed in [14]) are, in general, undecidable [1] (it should be noted, however, that fragments of the language with $\downarrow$ over Kripke frames are decidable and as such play a role, e.g., in conjunctive query answering in description logic [11]). As a consequence, completeness of pure extensions and local binding is the best we can hope for – it establishes recursive enumerability of the set of valid formulas, and it enables automated reasoning, if not decision procedures.

Specifically, we establish two separate criteria for the existence of named models. Although these criteria are (in all likelihood necessarily) less widely applicable than some previous coalgebraic results including those of [14], the generic results allow us to establish new completeness results for a wide variety of logics; in particular, we prove strong completeness of graded hybrid logic, and ultimately an extension of the description logic $\mathcal{SHOQ}$, with the $\downarrow$ binder over a wide variety of frame classes.

1. Coalgebraic Hybrid Logic

To make our treatment parametric in the syntax, we fix a modal similarity type Λ consisting of modal operators with associated arities throughout. For given countably infinite and disjoint sets P of propositional variables and N of nominals, the set $\mathcal{F}(\Lambda)$ of *hybrid Λ-formulas* is given by the grammar

$$\mathcal{F}(\Lambda) \ni \phi, \psi ::= p \mid i \mid \phi \wedge \psi \mid \neg\phi \mid \heartsuit(\phi_1, \ldots, \phi_n) \mid @_i\phi$$

where $p \in \mathsf{P}$, $i \in \mathsf{N}$ and $\heartsuit \in \Lambda$ is an n-ary modal operator. (Alternatively, we could regard *propositional variables* as nullary modal operators, thus avoiding their explicit mention altogether. We keep them explicit here, following standard practice in modal logic, as we have to deal with valuations anyway due to the presence of nominals.) We use the standard definitions for the other propositional connectives $\to, \leftrightarrow, \vee$. The set of nominals occurring in a formula ϕ is denoted by $\mathsf{N}(\phi)$, similarly for sets of formulas. A formula of the form $@_i\phi$ is called an *@-formula*. Semantically, nominals i denote individual states in a model, and $@_i\phi$ stipulates that ϕ holds at state i.

To reflect parametricity also semantically, we equip hybrid logics with a *coalgebraic semantics* extending the standard coalgebraic semantics of modal logics [16]: we fix throughout a Λ-*structure* consisting of an endofunctor $T : \mathsf{Set} \to \mathsf{Set}$ on the category of sets, together with an assignment of an *n-ary predicate lifting* $[\![\heartsuit]\!]$ to every n-ary modal operator $\heartsuit \in \Lambda$, i.e. a set-indexed family of mappings $([\![\heartsuit]\!]_X : \mathcal{P}(X)^n \to \mathcal{P}(TX))_{X \in \mathsf{Set}}$ that satisfies

$$[\![\heartsuit]\!]_X \circ (f^{-1})^n = (Tf)^{-1} \circ [\![\heartsuit]\!]_Y$$

for all $f : X \to Y$. In categorical terms, $[\![\heartsuit]\!]$ is a natural transformation $\mathcal{Q}^n \to \mathcal{Q} \circ T^{op}$ where $\mathcal{Q} : \mathsf{Set}^{op} \to \mathsf{Set}$ is the contravariant powerset functor.

In this setting, T-coalgebras play the roles of *frames*. A *T-coalgebra* is a pair (C, γ) where C is a set of *states* and $\gamma : C \to TC$ is the *transition function*. When γ is clear from the context, we refer to (C, γ) just as C. A *(hybrid) T-model* $M = (C, \gamma, V)$ consists of a T-coalgebra (C, γ) together with a *hybrid valuation* V, i.e. a map $\mathsf{P} \cup \mathsf{N} \to \mathcal{P}(C)$ that assigns singleton sets to all

nominals $i \in \mathsf{N}$. We say that M is *based* on the frame (C, γ). The singleton set $V(i)$ is tacitly identified with its unique element.

The semantics of $\mathcal{F}(\Lambda)$ is a satisfaction relation $\models$ between states $c \in C$ in hybrid T-models $M = (C, \gamma, V)$ and formulas $\phi \in \mathcal{F}(\Lambda)$, inductively defined as follows. For $x \in \mathsf{N} \cup \mathsf{P}$ and $i \in \mathsf{N}$,

$$M, c \models x \text{ iff } c \in V(x) \qquad \text{and} \qquad M, c \models @_i\phi \text{ iff } M, V(i) \models \phi.$$

Modal operators are interpreted using their associated predicate liftings, that is,

$$M, c \models \heartsuit(\phi_1, \ldots, \phi_n) \iff \gamma(c) \in [\![\heartsuit]\!]_C([\![\phi_1]\!]_M, \ldots, [\![\phi_n]\!]_M)$$

where $\heartsuit \in \Lambda$ is n-ary and $[\![\phi]\!]_M = \{c \in C \mid M, c \models \phi\}$ denotes the truth-set of ϕ relative to M. We write $M \models \phi$ if $M, c \models \phi$ for all $c \in C$. For a set $\Phi \subseteq \mathcal{F}(\Lambda)$ of formulas, we write $M, c \models \Phi$ if $M, c \models \phi$ for all $\phi \in \Phi$, and $M \models \Phi$ if $M \models \phi$ for all $\phi \in \Phi$. We say that Φ is *satisfiable* in a model M if there exists a state c in M such that $M, c \models \Phi$. If $\mathcal{A} \subseteq \mathcal{F}(\Lambda)$ is a set of axioms, also referred to as *frame conditions*, a frame (C, γ) is an *$\mathcal{A}$-frame* if $(C, \gamma, V) \models \phi$ for all hybrid valuations V and all $\phi \in \mathcal{A}$, and a model is an *$\mathcal{A}$-model* if it is based on an $\mathcal{A}$-frame. A frame condition is *pure* if it does not contain any propositional variables (it may however contain nominals). We recall notation from earlier work:

Notation 1. As usual, application of substitutions $\sigma : \mathsf{P} \to \mathcal{F}(\Lambda)$ to formulas ϕ is denoted $\phi\sigma$. For a set Σ of formulas and a set O of operators, we write $O\Sigma$ or $O(\Sigma)$ for the set of formulas arising by prefixing elements of Σ with an operator from O; e.g. $\Lambda(\Sigma) = \{\heartsuit(\phi_1, \ldots, \phi_n) \mid \heartsuit \in \Lambda \text{ } n\text{-ary}, \phi_1, \ldots, \phi_n \in \Sigma\}$ and $@\Sigma := \{@_i \mid i \in \mathsf{N}\}(\Sigma) = \{@_i\phi \mid i \in \mathsf{N}, \phi \in \Sigma\}$. Moreover, $\mathsf{Prop}(Z)$ denotes the set of propositional combinations of elements of some set Z. For $\phi \in \mathsf{Prop}(Z)$, we write $X, \tau \models \phi$ if ϕ evaluates to $\top$ in the boolean algebra $\mathcal{P}(X)$ under a valuation $\tau : Z \to \mathcal{P}(X)$. For $\psi \in \mathsf{Prop}(\Lambda(Z))$, the interpretation $[\![\psi]\!]_{TX,\tau}$ of ψ in the boolean algebra $\mathcal{P}(TX)$ under τ is the inductive extension of the assignment $[\![\heartsuit(p_1, \ldots, p_n)]\!]_{TX,\tau} = [\![\heartsuit]\!]_X(\tau(p_1), \ldots, \tau(p_n))$. We write $TX, \tau \models \psi$ if $[\![\psi]\!]_{TX,\tau} = TX$, and $t \models_{TX,\tau} \psi$ if $t \in [\![\psi]\!]_{TX,\tau}$. A set of formulas $\Xi \subseteq \mathsf{Prop}(\Lambda(Z))$ is *one-step satisfiable* w.r.t. τ if $\bigcap_{\phi \in \Xi} [\![\phi]\!]_{TX,\tau} \neq \emptyset$. We occasionally apply this notation to sets $Z \subseteq \mathcal{P}(X)$ with τ being just inclusion, in which case mention of τ is suppressed.

In the sequel, we will be interested in both *local* and *global* semantic consequence, where local consequence refers to satisfaction in a single state and global consequence to satisfaction in entire models. In fact, we consider local reasoning under global assumptions: given a set $\Phi \subseteq \mathcal{F}(\Lambda)$ of global assumptions (a *TBox* in description logic terminology) and a class $\mathcal{C}$ of models, we say that *ϕ is a local consequence of Ψ under global assumptions Φ for $\mathcal{C}$-models*, in symbols $\Phi; \Psi \models^{\mathcal{C}} \phi$, if for all $M \in \mathcal{C}$ such that $M \models \Phi$, $M, c \models \phi$ whenever $M, c \models \Psi$ (here, both Φ and Ψ are sets of arbitrary formulas, in particular not subject to any restrictions on the nesting depth of modal operators). The standard notions of local and global consequence are regained from this general definition by taking Φ or Ψ to be empty, respectively.

The distinguishing feature of the coalgebraic approach to hybrid and modal logics is the parametricity in both the logical language and the notion of frame: concrete instantiations of the general framework, in other words a choice of modal operators Λ and a Λ-structure T, capture the syntax and semantics of a wide range of modal logics, as witnessed by the following examples.

Examples 1.1. 1. The hybrid version of the modal logic K, *hybrid K* for short, has a single unary modal operator $\Box$, interpreted over the structure consisting of the powerset functor $\mathcal{P}$ (which takes a set X to its powerset $\mathcal{P}(X)$) and the predicate lifting $[\![\Box]\!]_X(A) = \{B \in \mathcal{P}(X) \mid B \subseteq A\}$. It is clear that $\mathcal{P}$-coalgebras $(C, \gamma : C \to \mathcal{P}(C))$ are in 1-1 correspondence with Kripke frames, and that the coalgebraic definition of satisfaction specializes to the usual semantics of the box operator.

2. *Graded hybrid logic* has modal operators $\Diamond_k$ 'in more than k successors, it holds that'. It is interpreted over the functor $\mathcal{B}$ that takes a set X to the set $\mathcal{B}(X) = X \to \mathbb{N} \cup \{\infty\}$ of multisets over X by $[\![\Diamond_k]\!]_X(A) = \{B \in \mathcal{B}(X) \mid \sum_{x \in A} B(x) > k\}$. This captures the semantics of graded modalities over *multigraphs* [8], which are precisely the $\mathcal{B}$-coalgebras. A more general set of operators is that of *Presburger logic* [9], which admits integer linear inequalities $\sum a_i \cdot \#(\phi_i) \geq k$ among formulas. Unlike in the purely modal case [19], hybrid multigraph semantics visibly differs from the more standard Kripke semantics of graded modalities, as the latter validates all formulas $\neg\Diamond_1 i$, $i \in \mathsf{N}$. However, both semantics agree if we additionally stipulate $\neg\Diamond_1 i$ as a global (pure) axiom. Thus, our completeness results for multigraph semantics derived below do transfer to Kripke semantics. In particular they apply to many description logics, which commonly feature both nominals and graded modal operators in the guise of *qualified number restrictions*.

3. *Hybrid CK*, the hybrid extension of the basic conditional logic CK, has a single binary modal operator $\Rightarrow$, written in infix notation. Hybrid CK is interpreted over the functor Cf that maps a set X to the set $\mathcal{P}(X) \to \mathcal{P}(X)$, whose coalgebras are selection function models [6], by putting $[\![\Rightarrow]\!]_X(A, B) = \{f : \mathcal{P}(X) \to \mathcal{P}(X) \mid f(A) \subseteq B\}$.

4. *Classical hybrid logic* (the hybrid version of the logic E of neighbourhood frames, referred to as (the minimal) classical modal logic in [6]) has a single, unary modal operator $\Box$ and is interpreted over *neighbourhood frames*, that is, coalgebras for the functor $NX = \mathcal{P}(\mathcal{P}(X))$ (more precisely, the double contravariant powerset functor). The semantics of classical modal logic is defined by the lifting $[\![\Box]\!]_X(A) = \{S \in NX \mid A \in S\}$. *Monotone hybrid logic* has the same similarity type, but is interpreted over upwards closed neighbourhood frames, or coalgebras for the functor $MX = \{S \in NX \mid S \text{ upwards closed}\}$ where upwards closure refers to subset inclusion.

5. The syntax of coalition logic over a set N of agents is given by the similarity type $\{[C] \mid C \subseteq N\}$, and the operator $[C]$ reads as "coalition C has a joint strategy to enforce ...". The formulas of (hybrid) coalition logic are interpreted over game frames, i.e., coalgebras for the functor

$$\mathsf{G}(X) = \{(f, (S_i)_{i \in N}) \mid \textstyle\prod_{i \in N} S_i \neq \emptyset, f : \prod_{i \in N} S_i \to X\}$$

(a class-valued functor, technically speaking, which however does not cause problems). The semantics arises via the liftings

$$[\![[C]]\!]_X(A) = \{(f, (S_i)_{i \in N}) \in \mathsf{G}(X) \mid \exists (s_i)_{i \in C} \forall (s_i)_{i \in N \setminus C} (f((s_i)_{i \in N}) \in A\}.$$

We proceed to present a Hilbert-style proof system for coalgebraic hybrid logics, which we prove to be sound and strongly complete. This requires that the logic at hand satisfies certain coherence conditions between the axiomatization and the semantics — in fact the *same* conditions as in the purely modal case, which are easily verified *local* properties that can be verified without reference to T-models and are already known to hold for a large variety of logics [16, 19].

Proof systems for coalgebraic logics are most conveniently described in terms of one-step rules, as follows.

Definition 1.2. A *one-step rule* over Λ is a rule ϕ/ψ where $\phi \in \mathsf{Prop}(\mathsf{P})$ and $\psi \in \mathsf{Prop}(\Lambda(\mathsf{P}))$ (in fact, ψ may be restricted to be a disjunctive clause, which however is not relevant here). The rule ϕ/ψ is *one-step sound* if $TX, \tau \models \psi$ whenever $X, \tau \models \phi$ for a valuation $\tau : \mathsf{P} \to \mathcal{P}(X)$. Given a set $\mathcal{R}$ of one-step rules and a valuation $\tau : \mathsf{P} \to \mathcal{P}(X)$, a set $\Xi \subseteq \mathsf{Prop}(\Lambda(\mathsf{P}))$ is *one-step consistent* [20] if the set $\Xi \cup \{\psi\sigma \mid \sigma : \mathsf{P} \to \mathsf{Prop}(\mathsf{P}); \phi/\psi \in \mathcal{R}; X, \tau \models \phi\sigma\}$ is propositionally consistent.

One-step sound rules are sound, and we will assume one-step soundness tacitly in the sequel. Completeness hinges on variants of the notion of one-step completeness [19], which we define further

below. As the notion of one-step rule does not involve hybrid features, suitable rule sets can just be inherited from the corresponding modal systems; for graded logics, conditional logics, and many others, such rule sets are found, e.g., in [22, 21]. We recall that the one-step complete rule set for (hybrid) K consists of the rules

$$\frac{a}{\Box a} \qquad \frac{a \wedge b \to c}{\Box a \wedge \Box b \to \Box c}\,.$$

A set $\mathcal{R}$ of one-step rules now gives rise to a Hilbert system $\mathcal{LR}$ by adjoining propositional tautologies and the hybrid axioms, and closing under modus ponens, rule application, and $@$-necessitation. Formally, we write $\Phi \vdash_{\mathcal{LR}} \phi$ for a set Φ of formulas, the *global assumptions* (or the *TBox*), and a formula ϕ if ϕ is contained in the smallest set that

- contains Φ and all instances of propositional tautologies
- contains all instances of $@$-introduction $i \wedge \phi \to @_i\phi$ and make-or-break

  $$(\mathrm{mob}) \quad @_i p \to (\heartsuit(q_1, \ldots, q_n) \leftrightarrow \heartsuit(@_i p \wedge q_1, \ldots, @_i p \wedge q_n))$$

  together with all instances of the axioms $\neg @_i \bot$, $\neg @_i \phi \leftrightarrow @_i \neg \phi$, $@_i(\phi \wedge \psi) \leftrightarrow (@_i\phi \wedge @_i\psi))$, $@_i i$, $@_i j \leftrightarrow @_j i$, $@_i k \wedge @_j p \to @_i p$; and
- is closed under instances of $@$-generalization $p/@_i p$, instances of rules in $\mathcal{R}$, and modus ponens.

The second group of axioms ensures that $i \sim j :\equiv @_i j$ defines an equivalence relation on nominals and that $@_i$ distributes over propositional connectives. The (mob) axiom captures the fact that the truth set of an $@$-formula is either empty or the whole model; in the case of hybrid K, it is equivalent to the standard back axiom $@_i\phi \to \Box @_i \phi$.

We write $\Phi; \Psi \vdash_{\mathcal{LR}} \phi$ if there are $\psi_1, \ldots, \psi_n \in \Psi$ such that $\Phi \vdash_{\mathcal{LR}} \psi_1 \wedge \cdots \wedge \psi_n \to \phi$. That is, $\Phi; \Psi \vdash_{\mathcal{LR}} \phi$ if there is a proof of ϕ from global assumptions Φ that additionally assumes Ψ *locally*. As we assume that all one-step rules in $\mathcal{R}$ are one-step sound, soundness for both local and global consequence is immediate: we have $\Phi; \Psi \models^{\mathcal{C}} \phi$ (for $\mathcal{C}$ the class of all models) whenever $\Phi; \Psi \vdash_{\mathcal{LR}} \phi$. In [14], a criterion has been given for $\mathcal{LR}$ to be *weakly complete*, i.e. complete for the case where both the TBox Φ and the set Ψ of local assumptions are empty. Here, we extend this result to combined strong *global* and strong *local* completeness, i.e. to cover both an arbitrary TBox and an arbitrary set of local assumptions, even if $\mathcal{LR}$ is extended with pure frame conditions and local binding.

2. Strong Completeness of Pure Extensions

Pure completeness is a celebrated result in hybrid logic [3, Chapter 7.3]. In a nutshell, adding pure axioms to an already complete proof system for the hybrid extension of the modal logic K (Example 1.1), one retains completeness with respect to the class of frames that satisfy the additional axioms. In contrast to arbitrary modal axioms, pure axioms do not contain propositional variables, and therefore define – in the classical setting of hybrid K – first-order frame conditions. Here, we show that the same theorem is valid for a much larger class of logics, namely all coalgebraic hybrid logics satisfying one of two suitable sets of conditions. For the sake of readability, we restrict the technical development (not the examples) to the case of unary operators from now on until Section 2.2.

Definition 2.1. If $\mathcal{A}$ is a set of pure formulas and $\mathcal{R}$ is a set of one-step rules, we write $\Phi; \Psi \vdash_{\mathcal{LRA}+\mathsf{Name}} \phi$ if there are $\psi_1, \ldots, \psi_n \in \Psi$ such that $\psi_1 \wedge \ldots \psi_n \to \phi$ is $\mathcal{LR}$-derivable

from assumptions in Φ where additionally all substitution instances of axioms in $\mathcal{A}$ and the rule

$$(\mathsf{Name})\frac{i \to \phi}{\phi}(i \notin \mathsf{N}(\phi))$$

may be used in deductions. As before, we write $\Phi \vdash_{\mathcal{LRA}+\mathsf{Name}} \phi$ if $\Phi; \emptyset \vdash_{\mathcal{LRA}+\mathsf{Name}} \phi$.

In the above system, the rule (Name') $@_i\phi/\phi$ ($i \notin \mathsf{N}(\phi)$ and the rule

$$(\mathsf{NameCong})\ \ \frac{@_j(\phi \leftrightarrow \psi)}{\heartsuit\phi \leftrightarrow \heartsuit\psi}\ (j \notin \mathsf{N}(\phi, \psi))$$

are derivable. The system is clearly sound for both global and local consequence over $\mathcal{A}$-models in the same sense that $\mathcal{LR}$ is sound over T-models.

Definition 2.2. Let $\mathcal{A} \subseteq \mathcal{F}(\Lambda)$ be a set of pure axioms, and let $\Phi \subseteq \mathcal{F}(\Lambda)$ be a TBox. A set $\Psi \subseteq \mathcal{F}(\Lambda)$ is $(\mathcal{LRA} + \mathsf{Name})$-$\Phi$-*inconsistent* if there are $\psi_1, \ldots, \psi_n \in \Psi$ such that $\Phi \vdash_{\mathcal{LRA}+\mathsf{Name}} \neg(\psi_1 \wedge \cdots \wedge \psi_n)$. Otherwise, Ψ is $(\mathcal{LRA} + \mathsf{Name})$-$\Phi$-*consistent*. A subset of $@\mathcal{F}(\Lambda)$, i.e. a set of $@$-formulas, is called an *ABox* (again borrowing terminology from description logic). A *maximally* $(\mathcal{LRA}+\mathsf{Name})$-$\Phi$-*consistent ABox* is a maximal element K among the $(\mathcal{LRA}+\mathsf{Name})$-$\Phi$-consistent ABoxes, ordered by inclusion. For such a K, we write $S_K = \{K_i \mid i \in \mathsf{N}\}$, where $K_i = \{\phi \in \mathcal{F}(\Lambda) \mid @_i\phi \in K\}$, and put $V_K(i) = \{K_i\} = \{K_j \in S_K \mid i \in K_j\}$.

For the construction of a named model, we now fix a maximally $(\mathcal{LRA}+\mathsf{Name})$-$\Phi$-consistent ABox K. Later, we will take K to be a maximally consistent extension of a given set Φ of formulas, where we may assume, thanks to the rule (Name'), that $\Phi \subseteq @\mathcal{F}(\Lambda)$. We note the following trivial facts:

Lemma 2.3. *We have $\psi\sigma \in K_i$ for all $\psi \in \mathcal{A}$ and all substitutions σ, and moreover $K \cup \Phi \subseteq K_i$.*

Our goal is the construction of named canonical models in the following sense:

Definition 2.4. A *named canonical K-model* is a model (S_K, γ, V_K) such that

$$\gamma(K_i) \in [\![\heartsuit]\!]\hat{\phi} \quad \text{iff} \quad \heartsuit\phi \in K_i$$

for every nominal i, where $\hat{\phi} = \{K_j \in S_K \mid \phi \in K_j\}$.

It is clear that named canonical models are countable, as there are only countably many nominals.

Lemma 2.5 (Truth lemma for named canonical models)**.** *If $M = (S_K, \gamma, V_K)$ is a named canonical K-model and ϕ is a hybrid formula, then for every $K_i \in S_K$,*

$$M, K_i \models \phi \quad \textit{iff} \quad \phi \in K_i.$$

Hence, $M \models \Phi$, and M is an $\mathcal{A}$-model.

The last clause of the truth lemma follows from Lemma 2.3, the crucial point being that satisfaction of all substitution instances of $\mathcal{A}$ implies frame satisfaction of $\mathcal{A}$ because every state in the model is denoted by some nominal. We now establish two criteria for the existence of named canonical models. The first criterion assumes a stronger form of one-step completeness than the second, which instead demands that the modalities are *bounded*.

2.1. Pure Completeness for Strongly One-Step Complete Logics

The construction of named models hinges on the following notion of pastedness, which assures that nominals interact correctly across the whole model. For the rest of the section, we fix a one-step complete rule set $\mathcal{R}$, a set $\mathcal{A}$ of pure axioms, and a set $\Phi \subseteq \mathcal{F}(\Lambda)$ of global assumptions, and we write 'consistent' instead of '$(\mathcal{LRA} + \mathsf{Name})$-$\Phi$–consistent'.

Definition 2.6. An ABox K is 0-*pasted* if whenever $@_j(\phi \leftrightarrow \psi) \in K$ for all nominals j, then $@_i(\heartsuit\phi \leftrightarrow \heartsuit\psi) \in K$ for all nominals i.

It is clear that K can induce a named model only if K is 0-pasted. The construction of pasted ABoxes requires a Henkin-like extension of the logical language by adding new nominals. *Generally, we denote by $\mathcal{F}(\Lambda)^+$ an extended language with countably many new nominals not appearing in $\mathcal{F}(\Lambda)$.* We note the fact (slightly glossed over in the literature) that this extension is conservative:

Lemma 2.7. *If $\Psi \subseteq \mathcal{F}(\Lambda)$ is consistent, then Ψ remains consistent in $\mathcal{F}(\Lambda)^+$.*

Lemma 2.8 (Extended Lindenbaum lemma for 0-Pasted Sets)**.** *If $\Psi \subseteq \mathcal{F}(\Lambda)$ is consistent, then there exists a 0-pasted maximally consistent ABox $K \subseteq @\mathcal{F}(\Lambda)^+$ and a nominal i in $\mathcal{F}(\Lambda)^+$ such that $@_i\Psi \subseteq K$.*

(The proof of the above version of the Lindenbaum lemma uses Lemma 2.7, and exploits the Name' rule to introduce the nominal i.) As we are aiming for strong completeness results, (weak) one-step completeness as employed in weak completeness proofs using *finite* models [14, 19] is no longer adequate. Accordingly, our first criterion assumes a stronger condition:

Definition 2.9. A rule set $\mathcal{R}$ is *strongly one-step complete* if for every set X, every one-step consistent subset of $\mathsf{Prop}(\Lambda(\mathcal{P}(X)))$ is one-step satisfiable.

Lemma 2.10 (Named existence lemma, Version 1)**.** *If K is 0-pasted and $\mathcal{R}$ is strongly one-step complete, then there exists a named canonical K-model.*

In summary, we have:

Theorem 2.11. *If $\mathcal{R}$ is strongly one-step complete, then every extension of $\mathcal{LR}$ by pure axioms is both globally and locally strongly complete over countable hybrid models when equipped with the* Name *rule. That is, if $\Phi, \Psi \subseteq \mathcal{F}(\Lambda)$ and $\phi \in \mathcal{F}(\Lambda)$, then $\Phi; \Psi \vdash_{\mathcal{LRA}+\mathsf{Name}} \phi$ whenever $\Phi; \Psi \models^{\mathcal{C}} \phi$, where $\mathcal{C}$ is the class of countable $\mathcal{A}$-models.*

Proof. As usual, we show that every $(\mathcal{LRA} + \mathsf{Name})$-$\Phi$-consistent set $\Psi \subseteq \mathcal{F}(\Lambda)$ is satisfiable in a countable $\mathcal{A}$-model M such that $M \models \Phi$ (where satisfiability is clearly invariant under passing from $\mathcal{F}(\Lambda)$ to $\mathcal{F}(\Lambda)^+$). The extended Lindenbaum lemma yields a 0-pasted maximally consistent subset ABox $K \subseteq \mathcal{F}(\Lambda)^+$ and a nominal i in $\mathcal{F}(\Lambda)^+$ such that $@_i\Psi \subseteq K$. By the named existence lemma, we find a named, hence countable, canonical K-model $M = (S_K, \gamma, V_K)$, and by the truth lemma (Lemma 2.5), M is an $\mathcal{A}$-model, $M \models \Phi$, and $M, K_i \models \Psi$. ∎

Remark 2.12. In the literature (e.g. [3, Theorem 7.29]), the above completeness theorem is sometimes phrased as "completeness with respect to named models", i.e. models where every state is the denotation of some nominal; such models also played a central role in the early development of hybrid logic by the Sofia school (see e.g. [15]). In detail, this means that every state of the model is the denotation of a nominal *in a language extended with countably many new nominals*. This extension is necessary, as otherwise the consistent set $\{\neg n \mid n \in \mathsf{N}\}$ would be satisfiable in a model where every state is named by a nominal $n \in \mathsf{N}$ of the original language, which is clearly

impossible. Completeness with respect to models where every state is named by a nominal in an extended language, on the other hand, is an immediate consequence of completeness with respect to countable models.

Example 2.13. The previous theorem establishes strong completeness results for pure extensions of all hybrid logics with neighbourhood semantics (Example 1.1.4) that are defined by rank-1 axioms [20], i.e. modal formulas where the nesting depth of modalities is uniformly equal to 1 (such as the monotonicity axiom $\Box(a \wedge b) \to \Box b$). For the monotonic cases, i.e. extensions of monotonic hybrid logic, these results are essentially known [24], while they seem to be new for the non-monotonic cases, i.e. extensions of classical hybrid logic not containing the monotonicity axiom, including, e.g., various deontic logics [12]. Moreover, the theorem newly proves strong completeness of the hybridization of coalition logic, as Theorem 3.2 of [17] essentially states that coalition logic satisfies strong one-step completeness.

2.2. Pure Completeness for Bounded Logics

The condition of strong one-step completeness used in the previous section is a comparatively rare phenomenon [20]; the strength of the condition becomes clear in the fact that, unlike in the classical case of Kripke semantics, the above did not require a notion of 1-pastedness [5]. We proceed to present an alternative approach for the case where one does have an analogue of the (Paste-1) rule — this is the case if the operators are *bounded*, i.e., their satisfaction hinges, in each case, on only finitely and boundedly many states of a model.

Definition 2.14. A modal operator $\heartsuit$ is *k-bounded* for $k \in \mathbb{N}$ with respect to a Λ-structure T if for every set X and every $A \subseteq X$,

$$[\![\heartsuit]\!]_X(A) = \bigcup_{B \subseteq A, \#B \leq k} [\![\heartsuit]\!]_X(B).$$

(This implies in particular that $\heartsuit$ is monotonic.) We say that Λ is bounded w.r.t. T if every modal operator $\heartsuit$ in Λ is $k_\heartsuit$-bounded for some $k_\heartsuit$.

The boundedness of an operator can now be internalized in the logical deduction system. In particular, for k-bounded operators $\heartsuit$, one has the *paste rule*

$$(\mathsf{Paste}\heartsuit(k)) \quad \frac{@_{j_1}\phi \wedge \dots \wedge @_{j_k}\phi \wedge @_i\heartsuit(j_1 \vee \dots \vee j_k) \to \psi}{@_i\heartsuit\phi \to \psi}$$

with the side condition that the j_r are pairwise distinct fresh nominals. We write $\Phi \vdash_{\mathcal{LRA}+\mathsf{Name}+\mathsf{Paste}} \phi$ if ϕ is derivable from assumptions in Φ in the system $\mathcal{LR} + \mathsf{Name}$ where additionally the rule $(\mathsf{Paste}\heartsuit(k))$ may be used in deductions for k-bounded operators $\heartsuit$. This induces the notion of $(\mathcal{LRA}+\mathsf{Name}+\mathsf{Paste})$-$\Phi$-consistency, which we briefly refer to as consistency as we fix Φ, $\mathcal{A}$, and $\mathcal{R}$ throughout. Again, the system is clearly sound, i.e. $\Phi; \Psi \models^{\mathcal{C}} \phi$ whenever $\Phi; \Psi \vdash_{\mathcal{LRA}+\mathsf{Name}+\mathsf{Paste}} \phi$, where $\mathcal{C}$ is the class of $\mathcal{A}$-models.

Examples 2.15. 1. *Hybrid K.* The modal operator $\Diamond$ is 1-bounded. The arising paste rule $(\mathsf{Paste}\Diamond(1))$ is precisely the rule $(paste\Diamond)$ of [4].

2. *Graded hybrid logic.* The modal operator $\Diamond_k$ is $(k+1)$-bounded. One thus has a paste rule

$$(\mathsf{Paste}\Diamond_k(k+1)) \quad \frac{@_{j_1}\phi \wedge \dots \wedge @_{j_{k+1}}\phi \wedge @_i\Diamond_k(j_1 \vee \dots \vee j_{k+1}) \to \psi}{@_i\Diamond_k\phi \to \psi}$$

with side conditions as before.

3. *Positive Presburger hybrid logic.* A Presburger operator $\sum a_i \cdot \#(\__i) \geq k$ (Example 1.1) is k-bounded if the a_i are positive. E.g., this still allows expressing the statement, generally believed to be valid in the German national football league, that a team that has at least 37 points will not be relegated: $3 \cdot \#\mathsf{win} + 1 \cdot \#\mathsf{draw} \geq 37 \to \neg\mathsf{relegated}$.

The generalized 1-pastedness condition for bounded operators is as follows.

Definition 2.16. Let Λ be bounded. An ABox K is 1-*pasted* if whenever $\heartsuit$ is k-bounded and $@_i\heartsuit\phi \in K$, then $\{@_{j_1}\phi, \ldots, @_{j_k}\phi, @_i\heartsuit(j_1 \vee \cdots \vee j_k)\} \subseteq K$ for some nominals $j_1, \ldots, j_k$.

Again, it is clear that if Λ is bounded, then K can induce a named model only of K is 1-pasted. It is easy to see that if $\mathcal{R}$ is one-step complete and Λ is bounded (in fact already if $\mathcal{R}$ derives monotony for every $\heartsuit \in \Lambda$), then every 1-pasted set is also 0-pasted (Definition 2.6).

Lemma 2.17 (Extended Lindenbaum lemma for 1-pasted sets)**.** *Let Λ be bounded. If $\Psi \subseteq \mathcal{F}(\Lambda)$ is consistent, then there exist a 1-pasted maximally consistent ABox $K \subseteq @\mathcal{F}(\Lambda)^+$ and a nominal i in $\mathcal{F}(\Lambda)^+$ such that $@_i\Psi \subseteq K$, where $\mathcal{F}(\Lambda)^+$ is as in Section 2.1.*

Bounded operators now allow us to use a weaker version of one-step completeness. Instead of requiring that all one-step consistent sets are one-step satisfiable, we may restrict to *finite extensions* of propositional variables.

Definition 2.18. We say that $\mathcal{R}$ is *strongly finitary one-step complete* if for every set X, every one-step consistent subset of $\mathsf{Prop}(\Lambda(\mathcal{P}_{fin}(X)))$ is one-step satisfiable.

Clearly, any strongly one-step complete rule set is also strongly finitary one-step complete, but the example of graded hybrid logic witnesses that the converse is not true. We note that the weaker criterion still fails for probabilistic logics due to inherent non-compactness [23]; probabilistic logics also fail to be bounded, as a given probability $p \in [0, 1]$ can be split into any number of summands. Together with boundedness, the above condition enables a second version of the named existence lemma.

Lemma 2.19 (Named existence lemma, Version 2)**.** *If Λ is bounded, $\mathcal{R}$ is strongly finitary one-step complete, and K is 1-pasted, then there exists a named canonical K-model.*

Summarizing the above, we have the following extended completeness result.

Theorem 2.20. *Let Λ be bounded, and let $\mathcal{R}$ be strongly finitary one-step complete. Then every extension of $\mathcal{LR}$ by pure axioms is globally and locally strongly complete over countable hybrid models when equipped with the* Name *and* Paste *rules. In other words, if $\Phi, \Psi \subseteq \mathcal{F}(\Lambda)$, $\phi \in \mathcal{F}(\Lambda)$, and $\mathcal{C}$ is the class of all countable $\mathcal{A}$-models, then $\Phi; \Psi \vdash_{\mathcal{LRA}+\mathsf{Name}+\mathsf{Paste}} \phi$ whenever $\Phi; \Psi \models^{\mathcal{C}} \phi$.*

The proof follows the same route via extended Lindenbaum lemma, existence lemma, and truth lemma as for Theorem 2.11.

Example 2.21. By Example 2.15 and the fact that the known complete axiomatizations of the associated modal logics are in fact strongly finitary one-step complete, the previous theorem proves completeness of pure extensions of hybrid K, graded hybrid logic, and positive Presburger hybrid logic. Except for the standard case of hybrid K, these results seem to be new. In particular, we obtain completeness of pure extensions of graded (or positive Presburger) hybrid logic defining the following frame classes in multigraph semantics:

- The class of *Kripke frames*, seen as the class of multigraphs where the transition multiplicity between two individual states is always at most 1, defined by the pure axiom $\neg\Diamond_1 i$.

- The class of *reflexive* multigraphs, defined by the pure axiom $i \to \Diamond_0 i$.
- The class of *transitive* multigraphs, defined by the pure axioms $\Diamond_0 \Diamond_n i \to \Diamond_n i$, $n \geq 0$.
- The class of *symmetric* multigraphs, i.e., those where the transition multiplicity from x to y always equals the one from y to x, which is defined by the pure axioms $i \wedge \Diamond_k j \to @_j \Diamond_k i$.

Other frame classes of interest, see e.g. [3, Section 7.3], can be characterized similarly by translating the corresponding frame conditions from Kripke to multigraph semantics.

2.3. The Mixed Case

In some cases, the two methods laid out in the preceding sections can be combined for modal operators with several arguments that adhere, in each of their arguments, to one of the respective sets of semantic conditions. For the sake of readability, we formulate this explicitly only for the mixed binary case with a single modal operator, i.e. we assume in this section that $\Lambda = \{\heartsuit\}$ with $\heartsuit$ binary; the generalization to arbitrary numbers of arguments, several modal operators etc. should be obvious, and essentially only requires more elaborate terminology and notation.

Definition 2.22. We say that $\mathcal{R}$ is *(strongly, strongly finitary) one-step complete* if every one-step consistent subset of $\mathsf{Prop}(\Lambda(\mathcal{P}(X) \times \mathcal{P}_{\mathit{fin}}(X)))$ is one-step satisfiable. Moreover, we say that $\heartsuit$ is *k-bounded in the second argument* for $k \in \mathbb{N}$ if for every set X and all $A, B \subseteq X$, $[\![\heartsuit]\!]_X(A, B) = \bigcup_{C \subseteq A, \#C \leq k} [\![\heartsuit]\!]_X(A, C)$.

In the same manner as for Theorems 2.11 and 2.20, we derive:

Theorem 2.23. *If $\mathcal{R}$ is (strongly, strongly finitary) one-step complete and $\heartsuit$ is k-bounded in the second argument, then every extension of $\mathcal{LR}$ by pure axioms is both locally and globally strongly complete over countable hybrid models when equipped with the appropriate* Name *and* Paste *rules.*

Example 2.24. Hybrid CK (Example 1.1) is easily seen to be (strongly, strongly finitary) one-step complete, and the operator $>$ defined from the conditional operator $\Rightarrow$ by $a > b :\leftrightarrow \neg(a \Rightarrow \neg b)$ is 1-bounded in the second argument. By the above, it follows that every pure extension of hybrid CK is strongly complete over countable hybrid selection function models. E.g. we may define the class of conditional frames where all expressible conditions induce transitive relations by pure axioms $(\phi > \phi > i) \to (\phi > i)$. Such frames satisfy also the dual axiom (using a propositional variable a) $(\phi \Rightarrow a) \to (\phi \Rightarrow (\phi \Rightarrow a))$, an axiom for duplicating conditional assumptions. Similar statements apply to a combination of graded and conditional logic (obtainable compositionally using the methods of [21]), which has operators of the form $a \Rightarrow_k b$ "if a, then one normally has more than k instances of b".

The semantics of conditional logics in general has complex ramifications, involving, e.g., preference orderings or systems of spheres (see, e.g., [10, 18]); application of our methods to conditional logics beyond CK is the subject of further investigation. We note that pure completeness of a hybrid extension of Lewis' logic of counterfactuals has been established recently [18].

3. Local Binding

We next investigate completeness of a stronger hybrid language that includes the $\downarrow$ binder, which binds a state variable to the current state. Concretely, we allow formulas of the form $\downarrow i.\, \phi$, wherein the nominal i is locally bound (for compactness of presentation, we give up the usual distinction between nominals and state variables). Given a modal similarity type Λ, we write $\mathcal{F}_{\downarrow}(\Lambda)$ for the

ensuing extension of $\mathcal{F}(\Lambda)$. The reading of the formula $\downarrow i.\phi$ is "ϕ holds for the current state i". The satisfaction relation in the extended logic is defined by an additional clause for the $\downarrow$ binder,

$$(C, \gamma, V) \models \downarrow i.\, \phi \text{ iff } (C, \gamma, V[c/i]) \models \phi$$

where c is a state in a coalgebra C and $V[c/i]$ is obtained from V by modifying the value of i to c. The semantics of the $\downarrow$ binder immediately translates into the axiom scheme (see e.g. [4])

$$(\mathsf{DA}) \qquad @_i((\downarrow j.\, \phi) \leftrightarrow \phi[i/j]).$$

Given a set $\mathcal{R}$ of Λ-rules, a set $\Phi \subseteq \mathcal{F}_{\downarrow}(\Lambda)$ of formulas and a set $\mathcal{A} \subseteq @\mathcal{F}_{\downarrow}(\Lambda)$ of pure axioms, we write $\Phi \vdash_{\mathcal{LRA}+\mathsf{Name}+\mathsf{Paste}+\mathsf{DA}} \phi$ for the extension of the associated provability predicate $\vdash_{\mathcal{LRA}+\mathsf{Name}+\mathsf{Paste}}$ with (DA). Using (DA), one easily proves an extension of the truth lemma for named models (Lemma 2.5) to $\mathcal{F}_{\downarrow}(\Lambda)$, so that the completeness results for pure extensions proved before (Theorems 2.11, 2.20, and 2.23) transfer immediately to $\mathcal{L}_{\downarrow}$. We make this explicit for the bounded case:

Theorem 3.1. *If Λ is bounded and $\mathcal{R}$ is strongly finitary one-step complete, then every pure extension of $\mathcal{L}_{\downarrow}$ is strongly locally and globally complete over countable hybrid models. In other words, $\Phi; \Psi \models^{\mathcal{C}} \phi$ iff $\Phi; \Psi \vdash_{\mathcal{LRA}+\mathsf{Name}+\mathsf{Paste}+\mathsf{DA}} \phi$ for all $\phi \in \mathcal{F}_{\downarrow}(\Lambda)$ and all $\Phi, \Psi \subseteq \mathcal{F}(\Lambda)$, where $\mathcal{C}$ is the class of all countable $\mathcal{A}$-models.*

Remark 3.2. As noted in [24], the named model construction more generally yields completeness for any *locally definable* extension of the hybrid language, i.e. any extension whose semantics *at named states* is defined by a formula similar to (DA).

Example 3.3. Continuing Example 2.15, Theorem 3.1 reproves not only the known completeness of pure extensions of hybrid K with $\downarrow$, but also the completeness of pure extensions of graded (or positive Presburger) hybrid logic with $\downarrow$. This extends easily to the multi-agent case, or, in description logic terminology, to description logics with multiple roles. As, moreover, both a role hierarchy and transitivity of roles can be defined using pure axioms, we thus arrive at a complete axiomatization of an extension of the description logic $\mathcal{SHOQ}$ with satisfaction operators and $\downarrow$, which has been used in connection with conjunctive query answering [11], and allows, e.g., talking about the number of stepchildren of a stepmother, in continuation of the stepmother example from [13], .

4. Conclusions

We have laid out two criteria for the existence of named canonical models in coalgebraic hybrid logics — one that applies to cases where one has an analogue of the so-called Paste-1 rule of standard hybrid logic, and one which applies to cases where one does not need any such rule. While the latter means essentially that the logic is equipped with a neighbourhood semantics, the former requires that all modal operators of the logic are bounded, i.e. there is always only a bounded number of states relevant for their satisfaction at each point. Our main novel example of this type is graded hybrid logic (and an extension of it using certain Presburger modalities [9]). The named model construction entails completeness of pure extensions and completeness of extended hybrid languages with the local binder $\downarrow$ (of which the I–me construct of [13] is a single-variable restriction), which we thus obtain as new results for, e.g., hybrid coalition logic, hybrid classical modal logic, several hybrid deontic logics, hybrid conditional logic, graded hybrid logic, and an extension of the description logic $\mathcal{SHOQ}$. An open question that remains is the existence of so-called orthodox axiomatizations [4] in the presence of $\downarrow$, as well as to find an analogue of the characterization result of [24] stating that a variant of the Paste-1 rule characterizes the Kripke models among the

topological models of $S4$. A further topic of investigation is to find decidable fragments of the language with $\downarrow$; we note slightly speculatively that the fragment used in [13] may, in our terminology, be seen as requiring that a suitably defined NNF of a formula contains only positive occurrences of bound nominals under bounded modal operators.

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