Description Logics and Fuzzy Probability

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Abstract

Uncertainty and vagueness are pervasive phenomena in real-life knowledge. They are supported in extended description logics that adapt classical description logics to deal with numerical probabilities or fuzzy truth values. While the two concepts are distinguished for good reasons, they combine in the notion of *probably*, which is ultimately a fuzzy qualification of probabilities. Here, we develop existing propositional logics of fuzzy probability into a full-blown description logic, and we show decidability of several variants of this logic under Łukasiewicz semantics. We obtain these results in a novel generic framework of *fuzzy coalgebraic logic*; this enables us to extend our results to logics that combine crisp ingredients including standard crisp roles and crisp numerical probabilities with fuzzy roles and fuzzy probabilities.

1 Introduction

Description logics (DLs) have emerged as a widely accepted and used knowledge representation framework. By-and-large, the semantics of DLs is based on relational structures [Baader *et al.*, 2003]. This provides a good fit for many applications, but needs to be extended if one requires concepts that involve degrees of vagueness. Extensions of this kind are commonly subsumed under the term *Fuzzy Description Logics* [Lukasiewicz and Straccia, 2008]. The prime example here is a logic of 'likes' where the assertion $\mathsf{likes}(a, b)$ may (semantically and syntactically) receive a fuzzy truth value, depending on the strength of affection.

Orthogonal to this, many knowledge representation formalisms have been extended with probabilities, e.g. propositional logics [Fagin *et al.*, 1990], DLs (overviews are in [Baader *et al.*, 2003; Lukasiewicz, 2008; Lutz and Schröder, 2010]), and first-order logics [Halpern, 1990; Jaeger, 2006]. Such logics make crisp statements about probabilities, e.g. that the incidence rate of a certain disease exceeds some numerical value. Semantically, this is accommodated by assigning probabilities to concepts that can then be compared, but in contrast to fuzzy logics, these comparisons are bivalent.

The fact that vagueness and probabilistic uncertainty are notions that deserve to be distinguished [Lukasiewicz and Straccia, 2008] does not preclude situations involving both. A point in case is the word *probably*, a vague qualification of a degree of uncertainty. In crisp logics, this phenomenon has been approached by either giving qualitative axiomatizations of likelihood [Burgess, 1969; Halpern and Rabin, 1987] or by imposing a threshold probability above which events are considered probable [Hamblin, 1959; Herzig, 2003]. In a *fuzzy* logic over probability distributions, one can define the truth value of $\mathit{probably}\ C$ for a fuzzy concept C as the expectation of C, read as a $[0, 1]$-valued random variable. This view goes back to Zadeh [1968] and is also taken in [Hájek, 2007], where a fuzzy propositional logic of $\mathit{probably}$ is developed and shown to be in PSPACE.

Hajek's reading of $\mathit{probably}$ is *global* in that it depends on a *single* probability distribution on the model. Syntactically, this is reflected in the fact that the $\mathit{probably}$ operator cannot be nested. Here, we introduce and analyse fuzzy probabilistic DLs that are interpreted *locally*, i.e. a distribution is associated to each point in the model, as in [Fagin and Halpern, 1994]. This allows modelling situations where, e.g., the probability of exhibiting a certain symptom depends on the disease. It elevates probability operators to the status of DL operators like existential restriction $\exists R$, which may be nested ad libitum, for the same reason: a local set of successors is associated to each point in the model.

While the logic of probably provides us with new expressive means for vague knowledge, it is clearly necessary to combine $\mathit{probably}$ with other DL connectives for meaningful applications. To accommodate this, we formulate our results in a more abstract framework that also allows treating fuzzy roles, such as likes, crisp roles, and quantitative uncertainty. We achieve this by using *coalgebraic logic*, which encapsulates the precise nature of knowledge operators as well as their interpretation, making our results applicable in a wide context by instantiating the more abstract framework.

We work with *Łukasiewicz* semantics for fuzzy connectives, which avoids some of the counter-intuitive properties of the simpler *minimalistic* (or *Zadeh*) fuzzy logic [Kundu and Chen, 1998]. Our main technical results are generic satisfiability algorithms that instantiate to yield the first algorithm for DLs with the $\mathit{probably}$ operator, overcoming difficulties related to unavailability of the finite model property for some reasoning problems; this can be combined modularly with a whole range of DL features involving crisp and fuzzy roles as well as numerical probabilities and nominals. Depending on details of the logic, our algorithms run in NEXPTIME or EXPSPACE, respectively, and in particular match the current best upper bound for Łukasiewicz fuzzy $\mathcal{ALC}$, NEXPTIME as implicit in [Straccia, 2005].

Related Work An overview of fuzzy and (quantitative) probabilistic description logics is given in [Lukasiewicz and Straccia, 2008]. Early research was focused on fuzzy DLs with Zadeh semantics, whose complexity is typically that of their classical counterparts; in fact, logical consequence remains mostly the same as well [Straccia, 2001; Bonatti and Tettamanzi, 2006; Stoilos *et al.*, 2007]. Fuzzy DLs with Łukasiewicz semantics are perceived to have better logical properties and have therefore received increased recent attention, e.g. [Hájek, 2005; Straccia, 2005; Bobillo and Straccia, 2011], but appear to have significantly higher complexity. For Łukasiewicz fuzzy $\mathcal{ALC}$ with general concept inclusions, even the finite model property fails [Bobillo *et al.*, 2010], so that we focus on acyclic TBoxes.

2 Fuzzy Description Logics by Example

Fuzzy description logics come in many different flavours. Roles may be fuzzy, crisp, or probabilistic, and operators come in various shapes and formats. All extend propositional logic, which we equip with Łukasiewicz semantics throughout this work (see Section 3). The syntax of the logics we consider here is then given by the grammar

$$\mathcal{F} \ni C, D ::= A \mid C \sqcap D \mid C \sqcup D \mid C \rhd D \mid \neg C \mid \heartsuit C$$

where A is an atomic concept, $\heartsuit \in \Lambda$ is a generic modal operator, and $\rhd$ is fuzzy implication. For readability, we focus on single roles here, the extension to multiple roles even with structurally different interpretations being straightforward.

We take the interpretation of a concept C to be a map $[\![C]\!] : X \to [0, 1]$ that assigns truth values to individuals in the carrier X of a model. If the structure underlying the model is crisp, we use *thresholds* to define the semantics of operators, given by

$$[\![C]\!]_\alpha = \{x \in X \mid [\![C]\!](x) \geq \alpha\}$$

so that $[\![C]\!]_\alpha$ is the set of those individuals that satisfy concept C with degree at least α.

Fuzzy $\mathcal{ALC}$ with crisp roles. The syntax of $\mathcal{ALC}$ arises via $\Lambda = \{\exists\}$, i.e. we use existential restriction as single operator. Models take the shape (X, ξ, π) where X is a set (of individuals), $\xi : X \to \mathcal{P}(X)$ determines the relational successors and π is a fuzzy valuation of atomic concepts. Given a concept C with interpretation $[\![C]\!] : X \to [0, 1]$, we put

$$[\![\exists C]\!](x) = \sup\{\alpha \mid \xi(x) \cap [\![C]\!]_\alpha \neq \emptyset\}$$

using the threshold notation introduced above. The primary purpose of this logic, called $\mathcal{ALC}_{\mathsf{L}}(C)$ in the sequel, is to import relational knowledge into a fuzzy setting.

Fuzzy $\mathcal{ALC}$. The syntax of fuzzy $\mathcal{ALC}$ is as above ($\exists$ is the only operator) but interpreted over fuzzy relations. Thus, models are of type (X, ξ, π) as above but $\xi : X \to (X \to [0, 1])$ is fuzzy and relates x and x' with degree $\xi(x)(x')$. The interpretation of $\exists$ now takes the form

$$[\![\exists C]\!](x) = \sup\{\xi(x)(y) \sqcap [\![C]\!](y) \mid y \in X\}$$

where $\sqcap$ is Łukasiewicz conjunction. It agrees with the standard semantics of fuzzy $\mathcal{ALC}$ [Lukasiewicz, 2008] (and is compatible ith the semantics of $\mathcal{ALC}_{\mathsf{L}}(C)$). We call this logic $\mathcal{ALC}_{\mathsf{L}}(F)$.

Quantitative fuzzy $\mathcal{ALC}$ is interpreted over local probability distributions and asserts likelihoods. We use operators $\mathbf{M}_p$, read 'with probability more than p', where $p \in [0, 1] \cap \mathbb{Q}$. Again, models take the form (X, ξ, π), but now $\xi : X \to \mathcal{D}(X)$ associates a discrete probability distribution $\xi(x)$ to each $x \in X$ (i.e. $\mathcal{D}(X) = \{\mu : X \to [0, 1] \mid \sum_{x \in X} \mu(x) = 1\}$). For a concept C we have

$$[\![\mathbf{M}_p C]\!](x) = \sup\{\alpha \mid \xi(x)([\![C]\!]_\alpha) > p\}$$

where $[\![C]\!] : X \to [0, 1]$. Informally, if the truth value of $\mathbf{M}_p C$ at x is α, then the local measure at x assigns probability $> p$ to the set individuals that satisfy C with degree $\geq \alpha$, and $\mathbf{M}_p$ picks the largest such α. The interpretation of $\mathbf{M}_p$ depends on the *local* probability distributions, thus, e.g., allowing us to model the mentioned fact that the probability of exhibiting a given symptom varies between diseases. E.g., we can express that encephalitis will show up as a headache with probability of more than 0.8 ($\mathsf{encephalitis} \rhd \mathbf{M}_{0.8}\mathsf{headache}$). We use $\mathcal{ALC}_{\mathsf{L}}(Q)$ to refer to this logic, and $\mathcal{ALC}_{\mathsf{L}}(Q_{\mathit{fin}})$ to denote the variant that is interpreted w.r.t. finitely supported probability distributions.

The logic of probably lifts the logic of fuzzy probability [Hájek, 2007] to a description logic context that allows arbitrary nesting of the probably operator. Syntactically, we have a single operator $\mathbf{P}$ (read 'probably'), interpreted over probability distributions as above, where

$$[\![\mathbf{P}C]\!](x) = \mathsf{E}_{\xi(x)}([\![C]\!]) = \textstyle\sum_{y \in X} [\![C]\!](y) \cdot \xi(x)(y);$$

i.e. $\mathbf{P}C$ is the expectation of C under the local distribution $\xi(y)$. Under a causal reading (different from the previous example), $\mathsf{headache} \rhd \mathbf{P}\mathsf{head\_trauma}$ asserts the vague probability judgement that an observed head trauma is probably the cause of a headache. We write $\mathcal{ALC}_{\mathsf{L}}(P)$ for the logic with the probably operator, and $\mathcal{ALC}_{\mathsf{L}}(P_{\mathit{fin}})$ to designate an interpretation over finitely supported distributions.

The logic of generally. This logic, similar to $\mathcal{ALC}_{\mathsf{L}}(P)$, has a single operator $\mathbf{G}$, read 'generally', or 'with high probability'. Again, models are of the form (X, ξ, π) where $\xi : X \to \mathcal{D}(X)$. The interpretation of $\mathbf{G}$ uses an explicit conversion function $h : [0, 1] \to [0, 1]$ (monotone, continuous, and piecewise linear) associating to a probability p the degree $h(p)$ to which p is 'high'. We stipulate that

$$[\![\mathbf{G}C]\!](x) = \sup\{\alpha \sqcap h(\xi(x)([\![C]\!]_\alpha))\}.$$

Here, α is a threshold value for membership in C, and the truth value of $\mathbf{G}C$ depends on both this threshold (the left conjunct) and the degree to which the likelihood of membership in C being $\geq \alpha$ is considered high (the right conjunct). In this logic, denoted by $\mathcal{ALC}_{\mathsf{L}}(G)$ (and $\mathcal{ALC}_{\mathsf{L}}(G_{\mathit{fin}})$ for its finitely supported semantics), we can express that headaches *generally* respond to analgesics.

Combinations and multiple roles. The heart of the semantics of all logics discussed above is coalgebraic, i.e. models are defined in terms of observations $X \to TX$ where T can be varied as needed. This makes it easy to combine features and roles: if roles R_i are interpreted w.r.t. structures of shape $X \to T_i X$ ($i = 1, 2$), then models $X \to T_1 X \times T_2 X$ interpret their combination by projecting to the respective component.

3 Preliminaries

To account for various different reasoning principles and compositionality, we parametrize our exposition *syntactically* over a set Λ of unary modal operators (higher arities are straightforward). The set $\mathcal{F}(\Lambda)$ of *concepts* is given by

$$\mathcal{F}(\Lambda) \ni C, D ::= A \mid C \sqcap D \mid C \sqcup D \mid \neg C \mid \heartsuit C$$

(with $C \rhd D := \neg C \sqcup D$), where $\heartsuit \in \Lambda$ and A is an atomic concept. The *size* $|C|$ of a concept C counts the number of logical operators and atomic concepts in C. As we can emulate atomic concepts by modal operators that ignore their argument, we omit them in the following. *Substitutions* are maps $\sigma : V \to \mathcal{F}(\Lambda)$ for a typically finite set V of variables; the size of σ is $|\sigma| = \sum_{v \in V} |\sigma(v)|$.

The *semantics* is then determined by the following data. Firstly, we fix the underlying type of *structures* by choosing a *set functor* T, i.e. a construction that assigns to each set X a set TX of *clusters* over X, and to each map $f : X \to Y$ a map $Tf : TX \to TY$, preserving identities and composition. Concepts are interpreted over (coalgebraic) T-*models* $M = (X, \xi)$ consisting of a set X of *states* and a map $\xi : X \to TX$ assigning to each state x a cluster $\xi(x)$ of successor states. E.g., if $TX = [0,1]^X \times [0,1]^{\mathsf{At}}$ where At is a set of atomic concepts, then T-models are fuzzy Kripke models. Second, all operators $\heartsuit \in \Lambda$ are assigned a *fuzzy predicate lifting* $[\![\heartsuit]\!]$, i.e. a family of maps

$$[\![\heartsuit]\!]_X : [0,1]^X \to [0,1]^{TX} \qquad \text{for all sets } X$$

that lifts fuzzy subsets of X to fuzzy subsets of TX, subject to the naturality condition $[\![\heartsuit]\!]_X(A \circ f) = [\![\heartsuit]\!]_Y(A) \circ Tf$ for $f : X \to Y$, $A \in [0,1]^Y$. The extension $[\![C]\!]_M : X \to [0,1]$ of a concept C in $M = (X, \xi, \pi)$ is defined recursively by pointwise application of the underlying propositional connectives according to Łukasiewicz semantics

$$[\![C \sqcap D]\!](x) = \max\{0, [\![C]\!](x) + [\![D]\!](x) - 1\}$$
$$[\![C \sqcup D]\!](x) = \min\{1, [\![C]\!](x) + [\![D]\!](x)\}$$
$$[\![\neg C]\!](x) = 1 - [\![C]\!](x)$$

and the clause

$$[\![\heartsuit C]\!] = [\![\heartsuit]\!]_X([\![C]\!]_M) \circ \xi.$$

We fix Λ, T, and the assignment of liftings throughout, and refer to these data collectively as a *logic* $\mathcal{L}$.

Example 3.1 (Fuzzy coalgebraic description logics)**.** All examples presented in Section 2 and their modular combinations can be expressed in the coalgebraic framework. E.g. for the *probably*-operator, $TX = \mathcal{D}(X)$ as in Section 2, and we have the lifting

$$[\![\mathbf{P}]\!]_X(A)(\mu) = \textstyle\sum_{x \in X} A(x) \cdot \mu(x)$$

that recovers the semantics introduced in Section 2.

We are concerned with *satisfiability* problem for DLs, which comes in a number of variants (that may lead to different complexity classes [Bonatti and Tettamanzi, 2006]) depending on how we deal with truth values.

Definition 3.2 (Constraints)**.** A *comparison operator* is one of $=, <, >, \leq, \geq$. A *constraint* $\sigma \bowtie \kappa$ consists of a substitution $\sigma : V \to \mathcal{F}(\Lambda)$, a valuation $\kappa : V \to [0,1]$, and a comparison operator $\bowtie$. We say that $\sigma \bowtie \kappa$ is *satisfiable* if there exists a T-model $M = (X, \xi)$ and a state $x \in X$ such that for all $v \in V$, $[\![\sigma(v)]\!]_M(x) \bowtie \kappa(v)$. The $\bowtie$-*satisfiability problem* is to decide whether a constraint $\sigma \bowtie \kappa$ is satisfiable.

Note that a constraint $\sigma \bowtie \kappa$ is essentially a conjunction $\bigwedge_{v \in V} \sigma(v) \bowtie \kappa(v)$. Evidently, $\bowtie$-satisfiability reduces to $\leq$-satisfiability for $\bowtie \in \{=, \geq\}$, and $>$-satisfiability reduces to $<$-satisfiability. None of $\leq$-satisfiability and $<$-satisfiability seem to reduce to the other. Note that $<$-satisfiability subsumes the dual of the *validity problem*, i.e. to decide whether a formula is always satisfied with truth degree 1.

4 Generic Closed Interval Satisfiability

We now develop generic reasoning procedures for coalgebraic fuzzy description logics, which we mainly instantiate to fuzzy probability. A substantial technical role is played by the distinction between closed and open truth degree intervals, i.e. $<$-satisfiability and $\leq$-satisfiability. Our generic algorithms are of high complexity (NEXPTIME and EXPSPACE) but match existing algorithms for fuzzy $\mathcal{ALC}$, which produce exponential-size mixed integer linear programming problems [Straccia, 2005]. The model constructions that witness correctness of our algorithms remove operators layer-by-layer, leading to the following notion of decomposition.

Definition 4.1 (Top-level decomposition)**.** A *top-level decomposition* of a substitution $\sigma : V \to \mathcal{F}(\Lambda)$ is a decomposition $\sigma = \sigma^\sharp \sigma^\flat$ where $\sigma^\flat : W \to \mathcal{F}(\Lambda)$, $\sigma^\sharp : V \to \mathsf{Prop}(\Lambda(W))$, and every variable in W occurs exactly once in $\sigma^\sharp$. Here, $\mathsf{Prop}(W)$ denotes propositional combinations of elements of W and $\Lambda(W) = \{\heartsuit w \mid w \in W\}$ are operator-prefixed formulas over W. This determines $\sigma^\sharp, \sigma^\flat$ uniquely up to renaming the variables in W.

In other words, the arguments of the top-most modal operators in σ are replaced with variables in $\sigma^\sharp$, and $\sigma^\flat$ records which formulas these variables stand for.

Definition 4.2 (Theory of a substitution)**.** Let $\sigma : V \to \mathcal{F}(\Lambda)$ be a substitution. The *theory* of σ is the set

$$\mathsf{Th}(\sigma) = \{\kappa : V \to [0,1] \mid \sigma = \kappa \text{ satisfiable}\}.$$

That is, $\mathsf{Th}(\sigma)$ records the possible joint truth values that the formulas $\sigma(v)$ can attain.

Definition 4.3 (Local constraints and models)**.** A *local constraint* $\Gamma = (\gamma, \sigma \bowtie \kappa)$ over sets V, W of variables consists of a set $\gamma \subseteq (V \to [0,1])$ of valuations for V, a substitution $\sigma : W \to \mathsf{Prop}(\Lambda(V))$, a valuation $\kappa : W \to [0,1]$, and a comparison operator $\bowtie$. A *local model* $M = (X, \tau, t)$ over V consists of a set X, a valuation $\tau : V \to (X \to [0,1])$, and a cluster $t \in TX$. We say that Γ is *satisfiable* if there exists M as above such that $M \models \Gamma$ in the sense that for all

$x \in X$, $\tau^T(x) \in \gamma$, where $\tau^T(x)(v) = \tau(v)(x)$, and for all $w \in W$, $[\![\sigma(w)]\!]_M \bowtie \kappa(w)$. Here, evaluation $[\![\phi]\!]_M$ of $\phi \in \mathsf{Prop}(\Lambda(V))$ over M is defined by extending the assignment

$$[\![\heartsuit a]\!]_M = [\![\heartsuit]\!](\tau(a))(t)$$

to (Łukasiewicz) propositional combinations.

Definition 4.4 (Local small model properties)**.** We say that $\mathcal{L}$ has the *local finite (polysize)* $\bowtie$*-model property* for a comparison operator $\bowtie$ if whenever a local constraint $(\gamma, \sigma \bowtie \kappa)$ is satisfiable, then it is satisfiable in a local model (X, τ, t) with X finite ($|X|$ polynomially bounded in $|\sigma|$).

Remark 4.5. It is clear that the local finite (polysize) $\bowtie$-model properties for $\bowtie \in \{=, \leq, \geq\}$ are equivalent, similarly for $\bowtie \in \{<, >\}$. Moreover, the local finite (polysize) $\leq$-model property implies the local finite (polysize) $<$-model property. It is unlikely that the converse holds; a possible counterexample is precisely $\mathcal{ALC}_{\mathsf{L}}(P)$ (Example 4.6). Moreover, despite the implication between the respective local small model properties, it does not seem to be the case that $<$-satisfiability can easily be reduced to $\leq$-satisfiability.

Example 4.6. It follows from results of [Hájek, 2007] that $\mathcal{ALC}_{\mathsf{L}}(P)$ has the local polysize $<$-model property. The proof relies on continuity arguments and on approximating infinite sums by finite partial sums. Essentially the same arguments work for the other probabilistic logics $\mathcal{ALC}_{\mathsf{L}}(Q)$, $\mathcal{ALC}_{\mathsf{L}}(G)$. No similar result is known for $\leq$ in place of $<$, even relaxing polysize to finite. Of course, the local finite $\leq$-model property holds trivially for any finitely branching logic, including the finitely branching probabilistic logics $\mathcal{ALC}_{\mathsf{L}}(X_{fin})$ for $X \in \{Q, P, G\}$; the local polysize $\leq$-model property then follows by results from linear programming as carried out for $\mathbf{P}$ in [Hájek, 2007].

Fuzzy $\mathcal{ALC}$ ($\mathcal{ALC}_{\mathsf{L}}(F)$) and $\mathcal{ALC}_{\mathsf{L}}(C)$ do have the local finite $\leq$-model property, which can be proved rather easily from the fact that solvability of systems $Ax \leq b$ of linear inequalities (for a matrix A and a vector b) is closed under infima in b; again, the polysize sharpening follows. (For $\mathcal{ALC}_{\mathsf{L}}(F)$, the local finite $\leq$-model property follows alternatively from results of [Hájek, 2005], which however employ heavy-weight methods from fuzzy first-order model theory.)

Theorem 4.7 (Local reduction)**.** *Let $\sigma : V \to \mathcal{F}(\Lambda)$ be a substitution with top-level decomposition $\sigma = \sigma^\sharp\sigma^\flat$, let $\bowtie \in \{=, <, >, \leq, \geq\}$, and let $\kappa : V \to [0, 1]$ be a valuation. Then $\sigma \bowtie \kappa$ is satisfiable iff the local constraint*

$$(\mathsf{Th}(\sigma^\flat), \sigma^\sharp \bowtie \kappa)$$

is satisfiable.

In the presence of the local finite $\leq$-model property, the local reduction theorem immediately implies a shallow tree model property. While in the classical case and in the very similar case of Zadeh logics [Straccia, 2001] the tree structure of models can often be exploited to obtain PSPACE decision procedures that explore one branch of the tree at time, this does not seem to be possible for Łukasiewicz logics, in which the branches are arithmetically entangled. We thus state only the arising exponential model property:

Corollary 4.8 (Exponential model property)**.** *Let $\mathcal{L}$ have the local polysize $\leq$-model property. Then every satisfiable constraint $\sigma \leq \kappa$ is satisfiable in a model with at most exponentially many states in $|\sigma|$.*

One then typically obtains a translation of the satisfiability problem into an exponential sized constraint in a suitable formalism, depending on the nature of the modalities:

Definition 4.9. We say that $\mathcal{L}$ is *polynomially existential first-order* if for every finite set X,

1. the set TX of clusters over X can be represented by a polynomial-sized existential first-order formula ϕ_X over the reals (i.e. $TX \cong \{(y_1, \dots, y_n) \mid \phi(y_1, \dots, y_n)\}$), and
2. for every $\heartsuit \in \Lambda$ and every comparison operator $\bowtie$, the formula $[\![\heartsuit]\!]_X(A)(t) \bowtie a$ is expressible as an existential first-order formula over the reals in the variables a and A_x, $x \in X$, the latter representing the truth values $A(x)$, and additional variables y_i describing $t \in TX$ according to 1), of polynomial size in $|X|$.

If, additionally, all atoms in the mentioned first-order formulas are linear inequalities, then $\mathcal{L}$ is *polynomially MILP*.

Example 4.10. The logic $\mathcal{ALC}_{\mathsf{L}}(P)$ is polynomially existential first-order: $\mu \in \mathcal{D}(X) \sqcap [\![\mathbf{P}]\!]_X(A)(\mu) = a$ is expressed by the existential (in fact, quantifier-free) first-order formula

$$\phi_X := \bigwedge_{x \in X} \mu_x \geq 0 \sqcap \sum_{x \in X} \mu_x = 1 \sqcap \sum_{x \in X} A_x \mu_x = a$$

involving variables a and A_x as in Definition 4.9 and variables μ_x representing probabilities $\mu(x)$; the size $|\phi_X|$ is clearly polynomial in $|X|$. The logics $\mathcal{ALC}_{\mathsf{L}}(X)$, for $X \in \{C, F, Q, G\}$, are even polynomially MILP.

Since mixed integer linear programming is in NP and the existential fragment of the first order logic of the reals is in PSPACE [Canny, 1988], we obtain

Corollary 4.11 (Complexity of $\leq$-satisfiability)**.** *Let $\mathcal{L}$ have the local polysize $\leq$-model property, and let $\bowtie \in \{<, \leq\}$. Then $\bowtie$-satisfiability is in* EXPSPACE *if $\mathcal{L}$ is polynomially existential first-order, and in* NEXPTIME *if $\mathcal{L}$ is polynomially MILP.*

Example 4.12. By the above, $\bowtie$-satisfiability in $\mathcal{ALC}_{\mathsf{L}}(C)$ and $\mathcal{ALC}_{\mathsf{L}}(F)$ is in NEXPTIME for $\bowtie \in \{<, \leq\}$. For fuzzy $\mathcal{ALC}$ ($\mathcal{ALC}_{\mathsf{L}}(F)$), this is exactly the complexity of the existing algorithms [Straccia, 2005]; we do not know of any matching lower bound. Moreover, $\bowtie$-satisfiability for $\bowtie \in \{<, \leq\}$ in the *finitely branching* probabilistic logics $\mathcal{ALC}_{\mathsf{L}}(X_{fin})$ is in EXPSPACE for $X = P$, and in NEXPTIME for $X \in \{Q, G\}$.

5 Generic Open Interval Reasoning

We now analyse the case where the local finite $\leq$-model property and hence Corollary 4.8 are not available, a case that we are particularly interested in as it includes the general forms of the probabilistic logics $\mathcal{ALC}_{\mathsf{L}}(X)$, $X \in \{Q, P, G\}$. The main reason for studying the infinitely branching case instead of just assuming finite branching is to ensure that restricting to the finite does not introduce artifacts into the mechanisms of logical consequence (see, e.g., [Schockaert *et al.*, 2009] for the effects of just restricting Łukasiewicz semantics to finitely many values).

In the absence of the local finite $\leq$-model property, we focus on $<$-satisfiability, which then brings topological and metric concepts into play [Hájek, 2007]. Recall that a function $f : X \to Y$ between metric spaces is *k-Lipschitz continuous* for $k \in \mathbb{R}$ if $d(f(x, y)) \leq kd(x, y)$ for all $x, y \in X$ (where we denote both metrics by d). It is one of the pleasant features of Łukasiewicz semantics that its operators are Lipschitz continuous (unlike for Gödel or product logic). For a set X, we regard the set $X \to [0, 1]$ as a metric space, equipped with the supremum metric.

Definition 5.1 (Lipschitz logics)**.** We say that $\mathcal{L}$ is *Lipschitz* if for every $\heartsuit \in \Lambda$ there exists $k_\heartsuit \in \mathbb{R}$ (*pspace* computable from $\heartsuit$) such that for every every $t \in TX$, the map $(X \to [0, 1]) \to V$, $A \mapsto [\![\heartsuit]\!]_X(A)(t)$ is $k_\heartsuit$-Lipschitz continuous.

(Computability of Lipschitz constants is usually not an actual issue, and in fact often all operators are 1-Lipschitz.)

Example 5.2. All logics of Section 2 are Lipschitz. To see this for $\mathcal{ALC}_{\mathsf{L}}(P)$, let $\mu \in \mathcal{D}(X)$ be a discrete probability distribution on a set X. We claim that $[\![P]\!]_X(A)(\mu)$ is 1-Lipschitz in A: We have $[\![P]\!]_X(A)(\mu) = \sum_{x \in X} \mu(x)A(x)$, so if $A, A' : X \to [0, 1]$ such that $d(A, A') < \varepsilon$, then $|[\![P]\!]_X(A)(\mu) - [\![P]\!]_X(A')(\mu)| < \sum_{x \in X} \mu(x)\varepsilon = \varepsilon$.

The following facts are particular for Łukasiewicz semantics.

Lemma 5.3. *Let $\mathcal{L}$ be Lipschitz. Then for every substitution $\sigma : V \to \mathsf{Prop}(\Lambda(W))$ and every $t \in TX$, the map $(W \to (X \to [0, 1])) \to (V \to [0, 1])$ sending τ to $\lambda a \in V.\, [\![\sigma(a)]\!]_\tau(t)$ is Lipschitz continuous.*

Lemma 5.4. *Let $\mathcal{L}$ be Lipschitz. Then $\mathsf{Th}(\sigma)$ is a compact subset of $V \to [0, 1]$ for every substitution $\sigma : V \to \mathcal{F}(\Lambda)$.*

Recall that given $\varepsilon > 0$ and a subset A of a metric space (X, d), $U_\epsilon(A) = \{x \in X \mid d(y, A) < \epsilon\}$, where $d(y, A) = \sup_{a \in A} d(y, a)$. By Lemma 5.4, local reduction leads back to $\leq$-satisfiability; we escape from this by

Theorem 5.5 (Local ε-reduction)**.** *Let $\mathcal{L}$ be Lipschitz and have the local finite $<$-model property, and let $\sigma : V \to \mathcal{F}(\Lambda)$ be a substitution, V finite. Then there exists k, pspace computable from σ, such that for all valuations $\kappa : V \to [0, 1]$, the constraint $\sigma < \kappa$ is satisfiable iff the local constraint*

$$(U_\varepsilon(\mathsf{Th}(\sigma^\flat)), \sigma^\sharp < \kappa - k\varepsilon)$$

is satisfiable for some $\varepsilon > 0$.

Proof sketch. Let k be the Lipschitz constant of $\sigma^\sharp$ (Lemma 5.3). Then 'only if' is trivial; 'if' relies on compactness (Lemma 5.4) and the local finite $<$-model property. □

Theorem 5.6. *Let $\mathcal{L}$ be Lipschitz and have the local polysize model property. Then $<$-satisfiability (and hence validity) is in* ExpSpace *if $\mathcal{L}$ is polynomially first order, and in* NExpTime *if $\mathcal{L}$ is polynomially MILP.*

Proof. By recursive translation of $\sigma < \kappa$ into an exponential size existential first-order formula over the reals; the upper bounds then follow as in Corollary 4.11. The core step in the translation is local ε-reduction, in which k is explicitly computed while ε is just existentially quantified. The recursive call is then based on the observation that in the notation of Theorem 5.5 and Definition 4.3, $\tau^T(x) \in U_\varepsilon(\mathsf{Th}(\sigma^\flat))$ is equivalent to satisfiability of the constraint $\sigma^\flat < \tau^T(x) + \varepsilon \wedge \sigma^\flat > \tau^T(x) - \varepsilon$. □

Example 5.7. By Theorem 5.6, $<$-satisfiability in the general forms of the probabilistic logics $\mathcal{ALC}_{\mathsf{L}}(X)$ is in ExpSpace for $X = P$, and in NExpTime for $X \in \{Q, G\}$, i.e. although $\mathbf{G}$ is intuitively similar to $\mathbf{P}$, it is computationally simpler.

Remark 5.8. The combination of two logics, discussed at the end of Section 2 inherits the respective conditions of Theorem 5.6 and Corollary 4.11 from its constituents so that the combination of two logics satisfying the relevant assumptions is decidable in NExpTime or ExpSpace, respectively.

Remark 5.9 (Reasoning with acyclic TBoxes)**.** As usual, it is unproblematic to deal with *acyclic TBoxes* by on-the-fly expansion [Lutz, 1999] without affecting complexity.

Satisfaction operators and ABox reasoning For the sake of readability, we have omitted ABox reasoning from the presentation so far. It is easy to extend our results to cover not only ABoxes, but even *nominals* $i, j, \ldots$ taken from a fixed set N, and *satisfaction operators* $@_i$, $i \in N$. Nominals are designated atomic concepts which are interpreted as crisp singletons, i.e. individual states in a model, which are then called *named states*. Concepts $@_iC$ evaluate to the truth value of C in the state i. We can then express *concept assertions* $C(i) \bowtie a$, where C is a concept, $i \in N$, and $a \in [0, 1]$, by conjuncts $@_iC \bowtie a$ in a constraint. By using nominals in concepts, we can express also *role assertions*; e.g., in fuzzy $\mathcal{ALC}$, $(@_i\exists R.\, j) \bowtie a$ is equivalent to $R(i, j) \geq a$ (and we can indeed also express role constraints of the form $R(i, j) \leq a$, which are excluded, e.g., in [Straccia, 2005]). We do not impose the unique name assumption; distinctness of nominals i, j is expressed by $(@_i\neg j) \geq 1$ or by $(@_i\neg j) > 0$.

The extension of our results to the arising *fuzzy coalgebraic hybrid logic* follows the lines of [Myers *et al.*, 2009]. This requires adapting the notion of local constraint and local model to accommodate nominals, and restructuring the local reduction theorem. Details are omitted for lack of space; we note only that where the truth values of subformulas of the target formula and the ABox at named states are nondeterministically guessed in the crisp setting, we instead introduce existentially quantified real variables in the fuzzy setting. With these modifications, the generic complexity results obtained so far (Corollary 4.11, Theorem 5.6) extend to the hybrid case, and their instantiation to our example logics requires only minor adaptation of the proofs of the local small model properties. Taking into account modularity (Remark 5.8), we obtain the following specific upper complexity bounds:

1. *Fuzzy $\mathcal{ALCO}$:* Reasoning with ABoxes and acyclic TBoxes in the fuzzy version of the description logic $\mathcal{ALCO}$ ($\mathcal{ALC}$ with nominals) is in NExpTime, even when satisfaction operators are included. To our knowledge, this logic was not previously known to be decidable (previous algorithms for fuzzy description logics with nominals [Bobillo

and Straccia, 2011] are limited to the finitely-valued version of Łukasiewicz semantics).

2. *Fuzzy probability:* The upper bound NEXPTIME remains valid if we extend fuzzy $\mathcal{ALCO}$ with crisp roles, quantitative probability operators $\mathbf{M}_p$, and the *generally* operator $\mathbf{G}$. If we add the *probably* operator $\mathbf{P}$, the upper bound jumps to EXPSPACE. Here, the comparison operators admissible in the ABox are the same as for satisfiability checking, i.e. unrestricted in the finitely branching case, and $>$, $<$ in the countably branching case.

6 Conclusion

We have shown decidability of fuzzy description logics with an operator *probably* and variations thereof, generalizing previous results on a propositional fuzzy logic of *probably* [Hájek, 2007] to a full-blown description logic featuring nested probability operators, ABoxes, acyclic TBoxes, and nominals, as well as crisp and fuzzy relational roles. The key tool here is an extension of the generic framework of *coalgebraic logic* [Myers *et al.*, 2009] to the fuzzy setting, which not only enables us to prove results that apply to whole ranges of logics at once, but also allows us to use modularity results in order to obtain results for combined logics that mix various modal operators for free. An important new technical aspect that is brought into play here is the use of metric concepts such as compactness and Lipschitz continuity.

Although no tight lower bounds are known, there appears to be a substantial hitch in computational complexity caused by the arithmetic character of Łukasiewicz semantics. In future research, we will investigate the prospect of optimized reasoning in Łukasiewicz description logics, e.g. using column generation [Klinov and Parsia, 2009] and Gröbner bases. Moreover, we intend to extend the range of reasoning services, in particular to top-k query answering [Lukasiewicz and Straccia, 2007], which would, e.g., determine which candidates are most *likely* to be suitable for a given job.

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