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Terminological Cycles and the Propositional μ-Calculus

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Abstract

We investigate terminological cycles in the terminological standard logic $\mathcal{ALC}$ with the only restriction that recursively defined concepts must occur in their definition positively. This restriction, called syntactic monotonicity, ensures the existence of least and greatest fixpoint models. It turns out that as far as syntactically monotone terminologies of $\mathcal{ALC}$ are concerned, the descriptive semantics as well as the least and greatest fixpoint semantics do not differ in the computational complexity of the corresponding subsumption relation. In fact, we prove that in each case subsumption is complete for deterministic exponential time. We then show that the expressive power of finite sets of syntactically monotone terminologies of $\mathcal{ALC}$ is the very same for the least and the greatest fixpoint semantics and, moreover, in both cases they are *strictly* stronger in expressive power than $\mathcal{ALC}$ augmented by regular role expressions. These results are obtained by a direct correspondence to the so-called propositional μ-calculus which allows to express least and greatest fixpoints explicitly. We propose $\mathcal{ALC}$ augmented by the fixpoint operators of the μ-calculus as a unifying framework for all three kinds of semantics.

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Contents

1 Introduction

Terminological logics or *concept languages* have been designed for the logical reconstruction and specification of knowledge representation systems derivating from KL-ONE such as BACK, CLASSIC, $\mathcal{KRIS}$, and LOOM.[1] These systems are able to represent dictionary-like definitions, and their task is to arrange these definitions in a hierarchy according to semantic relations like subsumption and equivalence. The dictionary-like definitions these systems are able to represent are usually called *concept introductions*. They attach a unique concept to a concept name, where concepts can be built up from the universal concept $\top$ and concept names by applying logical connectives as well as quantification over roles. The universal concept denotes the full domain, whereas concept names represent not further specified subsets of the domain. Roles denote arbitrary binary relations over the domain. A typical example for a concept introduction of this kind is the following one defining leaves as trees which do not have any branch:

$$Leaf \doteq Tree \sqcap \neg\exists branch{:}\top \qquad (1)$$

It is perfectly straightforward to state the meaning of such a concept introduction in set-theoretical terms. As usual, the semantics of concepts and concept introductions is given in terms of *interpretations* and *models*. An interpretation $\mathcal{I}$ with the *domain* $\Delta^{\mathcal{I}}$ maps $\top$ to $\Delta^{\mathcal{I}}$, each concept name CN to an arbitrary subset $CN^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$ and each role name RN to a binary relation $RN^{\mathcal{I}}$ over $\Delta^{\mathcal{I}}$. Moreover, the logical connectives $\sqcap$, $\sqcup$ and $\neg$ are interpreted as the corresponding set operations on $\Delta^{\mathcal{I}}$ and $\exists RN{:}$ and $\forall RN{:}$ represent the existential and universal quantification over $RN^{\mathcal{I}}$. The meaning of a concept introduction is then given by requiring that an interpretation is a model of $C \doteq D$ iff the interpretation maps C and D to the very same subset of the domain. In case of (1) this means that each model of (1) has to satisfy the following equation:

$$Leaf^{\mathcal{I}} = Tree^{\mathcal{I}} \cap \{d \in \Delta^{\mathcal{I}} : \textit{there is no } e \textit{ s.t. } \langle d, e\rangle \in branch^{\mathcal{I}}\}$$

There are also algorithms to compute the subsumption and equivalence relation with respect to finite sets of concept introductions of this kind [Schmidt-Schauß and Smolka, 1991]. Problems arise, however, when cyclic or recursive concept introductions come into the picture. It is very natural to define, for example, a tree recursively as a node having only trees as branches:

$$Tree \doteq Node \sqcap \forall branch{:}\, Tree \qquad (2)$$

[1] For a good overview of the 'KL-ONE family' the reader is referred to [Woods and Schmolze, 1992]. For KL-ONE itself see [Brachman and Schmolze, 1985].

The models of this cyclic concept introduction have to satisfy the following equation:

$$Tree^{\mathcal{I}} = \; Node^{\mathcal{I}} \cap \{d \in \Delta^{\mathcal{I}} : \textit{for all } e \textit{ s.t. } \langle d, e\rangle \in branch^{\mathcal{I}}, e \in Tree^{\mathcal{I}}\}$$

Unfortunately, such recursive equations do not have always unique solutions. Take, for instance, an interpretation $\mathcal{I}$ such that the domain of $\mathcal{I}$ is $\mathbb{N}$, the set of all natural numbers. Suppose $Node^{\mathcal{I}}$ is $\mathbb{N}$ and $branch^{\mathcal{I}}$ is the successor relation on $\mathbb{N}$. Such an interpretation is a model of (2) if it satisfies the following equation:

$$\begin{aligned} Tree^{\mathcal{I}} &= \mathbb{N} \cap \{n \in \mathbb{N} : \textit{for all } m \textit{ s.t. } m = n+1, m \in Tree^{\mathcal{I}}\} \\ &= \{n \in \mathbb{N} : n+1 \in Tree^{\mathcal{I}}\} \end{aligned}$$

The question, then, is whether all these models actually force every node to be a tree, i.e., whether $Node^{\mathcal{I}} \subseteq Tree^{\mathcal{I}}$. The recursive equation above does not tell us anything about this since it has two conflicting solutions, viz. one in which $Tree^{\mathcal{I}}$ is $\mathbb{N}$ and one in which it is the empty set. This gives rise to the question whether one of these models should be preferred or not. In fact, there is an ongoing discussion on which models match our intuition best.[2] But even if a semantics has been fixed, it is not clear at all how to obtain the corresponding inference algorithms, except for very small languages [Baader, 1990]. Concerning the semantics of terminological cycles, there are essentially three rivals to consider. First, simply allowing *all* models results in what Nebel [1990, Chapter 5.2.3] called *descriptive semantics*. The remaining two alternatives allow only models which are the least or greatest ones with respect to the denotation of the *defined* concept names (i.e., those concept names which appear on the left-hand side of a concept introduction). Nebel [1990, Chapter 5.2.2] called these models *least* and *greatest fixpoint models* respectively. The terms *least* and *greatest*, however, apply only to models which coincide on the interpretation of all role names and all *primitive* (i.e., undefined) concept names. The previously mentioned model which interprets *Tree* as the empty set is therefore a least fixpoint model of (2), whereas the other one interpreting *Tree* as $\mathbb{N}$ is a greatest fixpoint model.

The consequences of resorting to one of these alternatives can be clarified in some cases in terms of the reflexive-transitive closure R^* of a role. In the descriptive semantics, for instance, (2) expresses only a necessary condition for being a tree, viz. $Tree \sqsubseteq \forall branch^*{:}Node$, but it is not equivalent to it. In contrast to this, the greatest fixpoint semantics expresses not only a necessary condition, but also a sufficient one. In fact, Baader [1990, Theorem 4.3.1] showed that the greatest fixpoint semantics makes (2) *equivalent*

[2] Nebel [1990, 1991], Baader [1990], as well as Dionne *et al.* [1992] have been contributed to this discussion.

to the acyclic concept introduction $Tree \doteq \forall branch^*{:}Node$. For this reason, Baader claimed the greatest fixpoint semantics to come off best [Baader, 1990, page 626]. However, the least fixpoint semantics can express such quantification over $branch^*$ just as well. To see this, take the definition of a non-tree contrary to the one of a tree:

$$NonTree \;\doteq\; \neg Node \sqcup \exists branch{:}NonTree \tag{3}$$

That is to say, a non-tree is something which is either no node or which has some branch being a non-tree. In this case, only the *least* fixpoint semantics forces (3) to be equivalent to $NonTree \doteq \exists branch^*{:}\neg Node$. As $\exists branch^*{:}\neg Node$ is equivalent to $\neg\forall branch^*{:}Node$, the least fixpoint semantics of (3) expresses the very contrary of the greatest fixpoint semantics of (2). Anyway, the least fixpoint semantics seems to be more adequate for defining a tree as it excludes infinite chains of the role *branch*. In fact, we shall see that it forces (2) to be equivalent to the acyclic concept introduction $Tree \doteq (\forall branch^*{:}Node) \sqcap \neg\exists branch^\omega$, where the concept $\exists branch^\omega$ stipulates the existence of some infinite chain of the role *branch*. We thereby allow only acyclic structures of finite depth, which is clearly a necessary condition for being a tree.

It should be stressed that even though both (2) and (3) alone have least and greatest fixpoint models, neither $\{(2), NonTree \doteq \neg Tree\}$ nor $\{(3), Tree \doteq \neg NonTree\}$ have any least or greatest fixpoint model. This is due to the fact that no model can be a least or greatest one with respect to the denotation of a concept and its complement, unless the domain of the interpretation is empty. For instance, the interpretation over $\mathbb{N}$ as considered above is a model of $\{(2), NonTree \doteq \neg Tree\}$ if it satisfies the following two equations:

$$Tree^{\mathcal{I}} \;=\; \{n \in \mathbb{N} : n+1 \in Tree^{\mathcal{I}}\} \tag{4}$$

$$NonTree^{\mathcal{I}} \;=\; \mathbb{N} \setminus Tree^{\mathcal{I}} \tag{5}$$

These equations have exactly two solutions in common, namely one in which $Tree^{\mathcal{I}}$ is $\mathbb{N}$ and $NonTree^{\mathcal{I}}$ is the empty set, whereas in the second solution it is the other way around. Of course, neither solution is a least or greatest one with respect to both $Tree^{\mathcal{I}}$ and $NonTree^{\mathcal{I}}$. It seems to be counterintuitive that (2) alone has least and greatest fixpoint models, whereas $\{(2), NonTree \doteq \neg Tree\}$ does not have any. Not only that (2) alone has a least fixpoint model, but there is also a least fixpoint model of (2) which is a least fixpoint model of $NonTree \doteq \neg Tree$ as well. To see this, consider some least fixpoint model of (2). In such a model *Tree* denotes a certain set of objects. Now, with the denotation of *Tree* being fixed, there remains only one single model of $NonTree \doteq \neg Tree$, viz. the one in which *NonTree* is interpreted as the complement of the denotation of *Tree*. This model is therefore also a least fixpoint model of $NonTree \doteq \neg Tree$. The difference is that a least fixpoint model of $\{(2), NonTree \doteq \neg Tree\}$ is a solution of (4) and (5) which is the least one with respect to both $Tree^{\mathcal{I}}$ and $NonTree^{\mathcal{I}}$, while a least fixpoint model of (2) which is also a least fixpoint model of $NonTree \doteq \neg Tree$ is a least solution of (5) when $Tree^{\mathcal{I}}$ is taken to be a least solution of (4). The notion of least and greatest fixpoint semantics as considered in the terminological logics literature cannot tell these different situations apart. To overcome this deficiency, we introduce prefixes μ and ν to indicate that each model must be a least and a greatest fixpoint model respectively. The different situations above then can be distinguished by the fact that $\mu\{(2), NonTree \doteq \neg Tree\}$ does not have any model while $\{\mu\{(2)\}, \mu\{NonTree \doteq \neg Tree\}\}$ does have one. The former terminology will be called *least fixpoint terminology*, whereas the latter will be called *complex fixpoint terminologies*. The least and the greatest fixpoint terminologies of a complex fixpoint terminology can be thought of as self-contained *definitions*. Of course, an interpretation is a model of a complex fixpoint terminology iff it is a model of each least and greatest fixpoint terminology it contains. Having complex fixpoint terminologies at our disposal, we can even express the fact that the greatest fixpoint semantics of (2) expresses the very contrary of the least fixpoint semantics of (3):

$$\{\nu\{(2)\}, \mu\{(3)\}\} \models NonTree \doteq \neg Tree$$

But what is perhaps most important, it has been overlooked in the whole terminological logics literature that terminological cycles can be analyzed in terms of the well-investigated *propositional μ-calculus* in a perfectly straightforward way. The propositional μ-calculus is an extension of propositional multi-modal logic to reason about (concurrent) programs which has been proposed by Pratt [1981] and by Kozen [1983]. It allows to represent both least and greatest fixpoints explicitly by expressions corresponding to $\mu A.\{A \doteq Node \sqcap \forall branch{:}A\}$ and $\nu A.\{A \doteq Node \sqcap \forall branch{:}A\}$. For instance, the least fixpoint models of $Tree \doteq Node \sqcap \forall branch{:}Tree$ are exactly the models of the following acyclic concept introduction:

$$Tree \doteq \mu A.\{A \doteq Node \sqcap \forall branch{:}A\}$$

With the help of this correspondence, we shall determine the computational complexity and the expressive power of terminological cycles with respect to the least and greatest fixpoint semantics and we will even be able to obtain the corresponding inference algorithms. This correspondence also suggests to augment $\mathcal{ALC}$ by the least and greatest fixpoint operators of the μ-calculus in order to obtain a unifying framework for all three kinds of semantics.

2 The Terminological Logic $\mathcal{ALC}$

To begin with, we fix the syntax of the base expressions which can be used to form terminologies. We decided to take the terminological standard logic $\mathcal{ALC}$, investigated by Schmidt-Schauß and Smolka [1991] in their seminal paper. We did so because $\mathcal{ALC}$ is a well-known concept language which has been investigated thoroughly and in spite of its elegance, it is quite expressive. Last but no least, it not only has been investigated theoretically, but it is also the core language of the system $\mathcal{KRIS}$ [Baader and Hollunder, 1991].

Definition 1. Assume $\mathcal{N}$ is the union of two disjoint, countably infinite sets, $\mathcal{N}_\mathcal{C}$ and $\mathcal{N}_\mathcal{R}$, the elements of which are called **concept names** and **role names** respectively. The concepts of $\mathcal{ALC}$ are inductively defined as follows:

1. Each concept name, $\top$, and $\bot$ is a concept of $\mathcal{ALC}$.
2. If C and D are concepts of $\mathcal{ALC}$ and if RN is a role name, then $C \sqcap D$, $C \sqcup D$, $\neg C$, $\forall RN{:}C$ and $\exists RN{:}C$ are all concepts of $\mathcal{ALC}$.
3. These are all concepts of $\mathcal{ALC}$.

Of course, we may use parentheses to resolve ambiguities. The concepts of $\mathcal{ALC}$ are to be interpreted as subsets of some domain in such a way that concept names denote subsets the domain and $\top$, $\bot$, $\sqcap$, $\sqcup$, and $\neg$ are interpreted as the full domain, the empty set, the intersection, the union, and the set complement respectively. Concepts of the form $\exists R{:}C$ represent all those elements d of the domain for which there is at least one element e which is related to d by R such that e is an element of C. In contrast to this, $\forall R{:}C$ denotes all those elements d of the domain for which *all* elements related to d by R are elements of C.

Notation 1. Assume $r \subseteq \Delta \times \Delta$ is an arbitrary binary relation over Δ and assume $d \in \Delta$. Then $\boldsymbol{r(d)}$ is defined to be $\{d : \langle d, e\rangle \in r\}$.

Definition 2. An **interpretation** $\mathcal{I}$ is a tuple $\langle \Delta^\mathcal{I}, .^\mathcal{I}\rangle$, where $\Delta^\mathcal{I}$ is a set, called the **domain** of $\mathcal{I}$, and $.^\mathcal{I}$ is a function mapping concept names to subsets of $\Delta^\mathcal{I}$ and role names to binary relations on $\Delta^\mathcal{I}$ such that:

$$\begin{aligned}
\top^\mathcal{I} &= \Delta^\mathcal{I}\\
\bot^\mathcal{I} &= \emptyset\\
(C \sqcap D)^\mathcal{I} &= C^\mathcal{I} \cap D^\mathcal{I}\\
(C \sqcup D)^\mathcal{I} &= C^\mathcal{I} \cup D^\mathcal{I}\\
(\neg C)^\mathcal{I} &= \Delta^\mathcal{I} \setminus C^\mathcal{I}\\
(\forall R{:}C)^\mathcal{I} &= \{d \in \Delta^\mathcal{I} : R^\mathcal{I}(d) \subseteq C^\mathcal{I}\}\\
(\exists R{:}C)^\mathcal{I} &= \{d \in \Delta^\mathcal{I} : R^\mathcal{I}(d) \cap C^\mathcal{I} \neq \emptyset\}
\end{aligned}$$

A terminology is just a finite set of concept introductions each of which attaches a unique concept to a concept name.

Definition 3. Assume $\mathcal{L}$ is some set of concepts and assume C and D are elements of $\mathcal{L}$. Then $\boldsymbol{C \doteq D}$ is called **axiom** of $\mathcal{L}$ and it is a **concept introduction** of $\mathcal{L}$ if C is a concept name. Axioms of the form $C \doteq C \sqcap D$ are called **primitive** and are abbreviated with $\boldsymbol{C \sqsubseteq D}$. A **terminology** of $\mathcal{L}$ is a finite set $\mathcal{T}$ of concept introductions of $\mathcal{L}$ such that for every concept name CN there is at most one concept C such that $CN \doteq C$ is an element of $\mathcal{T}$.

In order to state the meaning of such terminologies, we have to specify their models. As usual, models are interpretations forcing something to hold. In case of terminologies, a model is simply an interpretation respecting every the concept introduction of the terminology in the sense that the left-hand side of the concept introduction must denote the very same subset of the domain as the right-hand side. As terminologies like $\{CN \doteq \neg CN\}$ should not have any model, the domain of an model is required to be nonempty.

Definition 4. An interpretation $\langle \Delta^{\mathcal{I}}, .^{\mathcal{I}} \rangle$ is a **model** of the axiom $C \doteq D$ iff $C^{\mathcal{I}} = D^{\mathcal{I}}$ and $\Delta^{\mathcal{I}}$ is not empty, and it is a **model** of a set of axioms iff it is a model of each axiom of the set.

Recall that $C \sqsubseteq D$ is treated as an abbreviation of $C \doteq C \sqcap D$. It should be stressed that the models of $C \sqsubseteq D$ are in fact exactly those interpretations in which the denotation of C is a subset of the denotation of D. That is, an interpretation $\langle \Delta^{\mathcal{I}}, .^{\mathcal{I}} \rangle$ is a model of $C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

Definition 5. Suppose $\mathcal{A} \cup \{C \doteq D\}$ is an arbitrary set of axioms. Then $\mathcal{A}$ is said to **entail** $C \doteq D$ iff every model of $\mathcal{A}$ is a model of $C \doteq D$ as well. In this case, we write $\boldsymbol{\mathcal{A} \models C \doteq D}$, possibly omitting the curly brackets of $\mathcal{A}$ and $\mathcal{A}$ altogether if it is the empty set. We say that D **subsumes** C with respect to $\mathcal{A}$ iff $\mathcal{A} \models C \sqsubseteq D$. Moreover, C and D are **equivalent** iff $\emptyset \models C \doteq D$, and a concept is **coherent** iff it is not equivalent to $\bot$.

Next, we shall define the notions of cyclicity and acyclicity of terminologies. As a matter of fact, we shall rarely make use of these notions. In a paper on terminological *cycles* these notions should be defined though.

Definition 6. Assume $\mathcal{T}$ is some terminology and assume $CN \doteq C$ and $CN' \doteq C'$ are two concept introductions. We say $CN \doteq C$ **depends** on $CN' \doteq C'$ and write $(CN \doteq C) \lhd (CN' \doteq C')$ iff CN' occurs in C. If $\lhd_{\mathcal{T}}$ denotes the transitive closure of $\lhd$ over $\mathcal{T}$, then $CN \doteq C$ and $CN' \doteq C'$ are defined to be **mutually dependent** within $\mathcal{T}$ iff both $(CN \doteq C) \lhd_{\mathcal{T}} (CN' \doteq C')$ and $(CN' \doteq C') \lhd_{\mathcal{T}} (CN \doteq C)$ holds. In this case, we write $(CN \doteq C) \bowtie_{\mathcal{T}} (CN' \doteq C')$.

The reader may check that $\bowtie_{\mathcal{T}}$ is always transitive as well as symmetric, but it is not necessarily reflexive.

Definition 7. A terminology $\mathcal{T}$ is **cyclic** iff it contains concept introductions which are mutually dependent within $\mathcal{T}$, otherwise it is **acyclic**.

It can easily be seen that for each *acyclic* terminology $\mathcal{T}$, all three kinds of semantics coincide, i.e., the models of $\mathcal{T}$ are exactly its least and greatest fixpoint models.

3 Syntactically Monotone Fixpoint Terminologies of $\mathcal{ALC}$

In the previous section, we considered solely arbitrary models of terminologies rather than least or greatest fixpoint models. We now introduce prefixes μ and ν to distinguish terminologies for which any model is allowed from those for which only least and greatest fixpoint models are allowed.

Definition 8. Assume $\mathcal{L}$ is some set of concepts and assume $\mathcal{T}$ is an arbitrary terminology $\mathcal{T}$ of $\mathcal{L}$. Then $\mu\mathcal{T}$ is called **least fixpoint terminology** of $\mathcal{L}$, whereas $\nu\mathcal{T}$ is called **greatest fixpoint terminology** of $\mathcal{L}$. Moreover, assume $\mathcal{T} = \bigcup_{i=1}^{n} \mathcal{T}_i$, where $\mathcal{T}_1, ..., \mathcal{T}_n$ have to be pairwise disjoint. If each σ_i $(1 \leq i \leq n)$ is either μ or ν, then $\{\sigma_1\mathcal{T}_1, ..., \sigma_n\mathcal{T}_n\}$ is a **complex fixpoint terminology** of $\mathcal{L}$.

This means roughly that a complex fixpoint terminology of $\mathcal{L}$ consists of a finite set of least and greatest fixpoint terminologies which do not have any defined concept in common. For example, the following complex fixpoint terminology of $\mathcal{ALC}$ consists of two least fixpoint terminologies of $\mathcal{ALC}$:

$$\{\mu\{\mathit{Tree} \doteq \mathit{Node} \sqcap \forall \mathit{branch}{:}\, \mathit{Tree}\}, \mu\{\mathit{NonTree} \doteq \neg\, \mathit{Tree}\}\}$$

The meaning of such a complex fixpoint terminology can be given straightforwardly in terms of models of least and greatest fixpoint terminologies. An interpretation is a model of a complex fixpoint terminology iff it is a model of each of its elements. In order to state the meaning of least and greatest fixpoint terminologies, $\mu\mathcal{T}$ and $\nu\mathcal{T}$, all models of $\mathcal{T}$ which coincide on the interpretation of the primitive concept and role names of $\mathcal{T}$ must be compared to each other, hence the following two definitions:

Definition 9. A concept name CN is **defined** in the terminology $\mathcal{T}$ iff there is a concept C such that $CN \doteq C$ is an element of $\mathcal{T}$, and a concept name or a role name is **primitive** in $\mathcal{T}$ iff it occurs in $\mathcal{T}$ but is not defined in $\mathcal{T}$. We denote with $\boldsymbol{prim}(\mathcal{T})$ the set of all concept and role names which are primitive in $\mathcal{T}$. All these notions are extended to least and greatest fixpoint terminologies as well as to complex fixpoint terminologies correspondingly.

For instance, the concept and role names which are primitive in the complex fixpoint terminology above are *Node* and *branch*, whereas both *Tree* and *NonTree* are defined in this complex fixpoint terminology.

Definition 10. Suppose $\mathcal{I} = \langle \Delta^{\mathcal{I}}, .^{\mathcal{I}} \rangle$ and $\mathcal{J} = \langle \Delta^{\mathcal{J}}, .^{\mathcal{J}} \rangle$ are two interpretations with $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}}$ and suppose N is some set of concept and role names. Then $\mathcal{J}$ is **N-compatible** with $\mathcal{I}$ iff for every $x \in N$, $x^{\mathcal{J}} = x^{\mathcal{I}}$.

Definition 11. Assume $\mathcal{T}$ is some terminology and assume $\mathcal{I} = \langle \Delta^{\mathcal{I}}, .^{\mathcal{I}} \rangle$ is an interpretation. Then $\mathcal{I}$ is a **least fixpoint model** of $\mathcal{T}$ iff it is a model of $\mathcal{T}$ and, additionally, for each other model $\langle \Delta^{\mathcal{I}}, .^{\mathcal{J}} \rangle$ of $\mathcal{T}$ which is $prim(\mathcal{T})$-compatible with $\mathcal{I}$, it holds for all CN which are defined in $\mathcal{T}$ that $CN^{\mathcal{I}} \subseteq CN^{\mathcal{J}}$. The **greatest fixpoint models** of $\mathcal{T}$ are defined correspondingly by requiring $CN^{\mathcal{I}} \supseteq CN^{\mathcal{J}}$ instead of $CN^{\mathcal{I}} \subseteq CN^{\mathcal{J}}$. Clearly, $\mathcal{I}$ is a **model** of $\mu\mathcal{T}$ iff it is a least fixpoint model of $\mathcal{T}$ and it is a **model** of $\nu\mathcal{T}$ iff it is a greatest fixpoint model of $\mathcal{T}$. If Γ is a complex fixpoint terminology and if $\mathcal{A}$ is a set of axioms, then $\mathcal{I}$ is a **model** of $\Gamma \cup \mathcal{A}$ iff it is a model of each element of $\Gamma \cup \mathcal{A}$.

The motivation of the notion of *fixpoint* model is the observation that a terminology $\mathcal{T}$ together with an interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, .^{\mathcal{I}} \rangle$ induces a n-ary function $\mathcal{T}_{\mathcal{I}}$ mapping $(2^{\Delta^{\mathcal{I}}})^n$ into $(2^{\Delta^{\mathcal{I}}})^n$ if we assume that $\mathcal{T}$ comprises exactly n concept introductions. Consider, for instance, the terminology $\{Tree \doteq Node \sqcap \forall branch{:}Tree\}$, henceforth abbreviated with $\mathcal{T}ree$. This terminology together with an interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, .^{\mathcal{I}} \rangle$ induces a function $\mathcal{T}ree_{\mathcal{I}}$ which maps subsets of $\Delta^{\mathcal{I}}$ to subsets of $\Delta^{\mathcal{I}}$. This function is defined for each $X \subseteq \Delta^{\mathcal{I}}$ as follows:

$$\mathcal{T}ree_{\mathcal{I}}(X) \stackrel{\text{def}}{=} Node^{\mathcal{I}} \cap \{d \in \Delta^{\mathcal{I}} : branch^{\mathcal{I}}(d) \subseteq X\}$$

In principle, the function $\mathcal{T}ree_{\mathcal{I}}$ can be defined using an interpretation $\mathcal{J} = \langle \Delta^{\mathcal{I}}, .^{\mathcal{J}} \rangle$ which is $prim(\mathcal{T}ree)$-compatible with $\mathcal{I}$ such that $Tree^{\mathcal{J}}$ is X. Then $\mathcal{T}ree_{\mathcal{I}}(X)$ can defined to be $(Node \sqcap \forall branch{:}Tree)^{\mathcal{J}}$. Recall that $prim(\mathcal{T}ree)$ is $\{Node, branch\}$. As $\mathcal{J}$ is $\{Node, branch\}$-compatible with $\mathcal{I}$, and as $Tree^{\mathcal{J}}$ is X, both definitions in fact yield the very same function:

$$\begin{aligned}\mathcal{T}ree_{\mathcal{I}}(X) &\stackrel{\text{def}}{=} (Node \sqcap \forall branch{:}Tree)^{\mathcal{J}} \\ &= Node^{\mathcal{J}} \cap \{d \in \Delta^{\mathcal{I}} : branch^{\mathcal{J}}(d) \subseteq Tree^{\mathcal{J}}\} \\ &= Node^{\mathcal{I}} \cap \{d \in \Delta^{\mathcal{I}} : branch^{\mathcal{I}}(d) \subseteq X\}\end{aligned}$$

Definition 12. Suppose $\mathcal{I} = \langle \Delta^{\mathcal{I}}, .^{\mathcal{I}} \rangle$ is an interpretation and suppose $\mathcal{T}$ is a syntactically monotone terminology of the form $\{CN_i \doteq C_i : 1 \leq i \leq n\}$, where $CN_1, ..., CN_n$ are ordered by some fixed total ordering on $\mathcal{N}_C$. The **function induced by** $\mathcal{T}$ and $\mathcal{I}$ is the function $\mathcal{T}_{\mathcal{I}} : (2^{\Delta^{\mathcal{I}}})^n \to (2^{\Delta^{\mathcal{I}}})^n$ defined as follows: Assume $X_1, ..., X_n$ are subsets of $\Delta^{\mathcal{I}}$ and assume $\langle \Delta^{\mathcal{I}}, .^{\mathcal{J}} \rangle$ is an arbitrary interpretation which is $prim(\mathcal{T})$-compatible with $\mathcal{I}$ such that $CN_i^{\mathcal{J}}$ is X_i, for all i $(1 \leq i \leq n)$. Then $\mathcal{T}_{\mathcal{I}}(X_1, ..., X_n)$ is $\langle C_1^{\mathcal{J}}, ..., C_n^{\mathcal{J}} \rangle$.

It should be stressed that—although this definition only requires $\langle \Delta^{\mathcal{I}}, .^{\mathcal{J}} \rangle$ to be an *arbitrary* interpretation which is $prim(\mathcal{T})$-compatible with $\mathcal{I}$ such that for all i $(1 \leq i \leq n)$, $CN_i^{\mathcal{J}}$ is X_i—each $C_i^{\mathcal{J}}$ $(1 \leq i \leq n)$ does *not* depend on which $.^{\mathcal{J}}$ is actually chosen. The reason for this is that the concept and role names appearing in C_i are among $prim(\mathcal{T}) \cup \{CN_1, ..., CN_n\}$, so that $C_i^{\mathcal{J}}$ depends only on the interpretation of these concept and role names. That means, $\mathcal{T}_{\mathcal{I}}(X_1, ..., X_n)$ is uniquely defined and, therefore, $\mathcal{T}_{\mathcal{I}}$ is actually a *function*.

As usual, the fixpoints of the function $\mathcal{T}ree_{\mathcal{I}}$ are those subsets X of $\Delta^{\mathcal{I}}$ such that $\mathcal{T}ree_{\mathcal{I}}(X)$ is X. Take, for instance, the interpretation $\mathcal{I} = \langle \mathbb{N}, .^{\mathcal{I}} \rangle$ considered in the introduction which interprets *Node* as $\mathbb{N}$ and *branch* as the successor relation on $\mathbb{N}$. In this case, the function $\mathcal{T}ree_{\mathcal{I}}$ induced by $\mathcal{T}ree$ and $\mathcal{I}$ maps each subset X of $\mathbb{N}$ to the set of all natural numbers whose successors are in X:

$$\begin{aligned} \mathcal{T}ree_{\mathcal{I}}(X) &= Node^{\mathcal{I}} \cap \{n \in \mathbb{N} : branch^{\mathcal{I}}(n) \subseteq X\} \\ &= \mathbb{N} \cap \{n \in \mathbb{N} : \{n+1\} \subseteq X\} \\ &= \{n \in \mathbb{N} : n+1 \in X\} \end{aligned}$$

The fixpoints of this function are clearly $\emptyset$ and $\mathbb{N}$, so that $\emptyset$ is its least fixpoint, whereas $\mathbb{N}$ is its greatest.

Definition 13. Assume Δ is an arbitrary set and f is some n-ary function mapping $(2^{\Delta})^n$ to $(2^{\Delta})^n$. An element $\langle X_1, ..., X_n \rangle$ of $(2^{\Delta})^n$ is called **fixpoint** of f iff $f(X_1, ..., X_n)$ is $\langle X_1, ..., X_n \rangle$, and such a fixpoint is a **least fixpoint** of f iff for each other fixpoint $\langle Y_1, ..., Y_n \rangle$ of f and for all i $(1 \leq i \leq n)$, $X_i \subseteq Y_i$. The **greatest fixpoints** of f are defined correspondingly by requiring $X_i \supseteq Y_i$ instead of $X_i \subseteq Y_i$.

It can easily be seen that an arbitrary interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, .^{\mathcal{I}} \rangle$ is a model of the terminology $\mathcal{T}ree$ iff $Tree^{\mathcal{I}}$ is a fixpoint of the function $\mathcal{T}ree_{\mathcal{I}}$ induced by $\mathcal{T}ree$ and $\mathcal{I}$, i.e., $\mathcal{T}ree_{\mathcal{I}}(Tree^{\mathcal{I}})$ is $Tree^{\mathcal{I}}$. This implies directly that there is also an intimate relationship between the least and greatest fixpoint models $\mathcal{I}$ of the terminology $\mathcal{T}ree$ on the one hand and the least and greatest fixpoints of the function induced by $\mathcal{T}ree$ and $\mathcal{I}$ on the other hand. In fact, $\mathcal{I}$ is a model of the least fixpoint terminology $\mu\mathcal{T}ree$ iff $Tree^{\mathcal{I}}$ is the least fixpoint of the function induced by $\mathcal{T}ree$ and $\mathcal{I}$. Obviously, the corresponding statement holds for $\nu\mathcal{T}ree$ as well.

Proposition 1. *Suppose $\mathcal{I} = \langle \Delta^{\mathcal{I}}, .^{\mathcal{I}} \rangle$ is an arbitrary interpretation, $\mathcal{T}$ is some terminology, and $\mathcal{T}_{\mathcal{I}}$ is the function induced by $\mathcal{T}$ and $\mathcal{I}$. Assume,* *moreover, $\mathcal{T}$ is of the form $\{CN_i \doteq C_i : 1 \leq i \leq n\}$, where $CN_1, ..., CN_n$ are ordered by the same ordering as in the definition of $\mathcal{T}_\mathcal{I}$. Then $\mathcal{I}$ is a model of $\mu\mathcal{T}$ iff $\langle CN_1^\mathcal{I}, ..., CN_n^\mathcal{I}\rangle$ is the least fixpoint of $\mathcal{T}_\mathcal{I}$, and it is a model of $\nu\mathcal{T}$ iff $\langle CN_1^\mathcal{I}, ..., CN_n^\mathcal{I}\rangle$ is the greatest fixpoint of $\mathcal{T}_\mathcal{I}$.*

Proof. It suffices to prove that $\mathcal{I}$ is model of $\mathcal{T}$ iff $\langle CN_1^\mathcal{I}, ..., CN_n^\mathcal{I}\rangle$ is a fixpoint of $\mathcal{T}_\mathcal{I}$. Assume $\langle CN_1^\mathcal{I}, ..., CN_n^\mathcal{I}\rangle$ is an arbitrary fixpoint of $\mathcal{T}_\mathcal{I}$, i.e.,

$$\mathcal{T}_\mathcal{I}(CN_1^\mathcal{I}, ..., CN_n^\mathcal{I}) = \langle CN_1^\mathcal{I}, ..., CN_n^\mathcal{I}\rangle$$

Now, take an arbitrary interpretation $\langle \Delta^\mathcal{I}, .^\mathcal{J}\rangle$ which is $prim(\mathcal{T})$-compatible with $\mathcal{I}$ such that for all i $(1 \leq i \leq n)$, $CN_i^\mathcal{J}$ is $CN_i^\mathcal{I}$. Then $\mathcal{T}_\mathcal{I}(CN_1^\mathcal{I}, ..., CN_n^\mathcal{I})$ is defined as follows:

$$\mathcal{T}_\mathcal{I}(CN_1^\mathcal{I}, ..., CN_n^\mathcal{I}) \stackrel{\text{def}}{=} \langle C_1^\mathcal{J}, ..., C_n^\mathcal{J}\rangle$$

As for all i $(1 \leq i \leq n)$, $CN_i^\mathcal{J}$ is $CN_i^\mathcal{I}$, $\mathcal{J}$ is not only $prim(\mathcal{T})$-compatible with $\mathcal{I}$, but also $\{CN_1, ..., CN_n\}$-compatible with $\mathcal{I}$. This means, for all i with $1 \leq i \leq n$, $C_i^\mathcal{J}$ and $C_i^\mathcal{I}$ actually do not differ, since the concept and role names occurring in $C_1, ..., C_n$ are among $prim(\mathcal{T}) \cup \{CN_1, ..., CN_n\}$. This together with the two equations above yields the following:

$$\mathcal{T}_\mathcal{I}(CN_1^\mathcal{I}, ..., CN_n^\mathcal{I}) = \langle CN_1^\mathcal{I}, ..., CN_n^\mathcal{I}\rangle = \langle C_1^\mathcal{I}, ..., C_n^\mathcal{I}\rangle$$

But this is just to say that $\mathcal{I}$ is a model of $\mathcal{T}$. □

To ensure the existence of least and greatest fixpoint models of a terminology $\mathcal{T}$, the function induced by $\mathcal{T}$ and an arbitrary interpretation should be monotonically increasing (or monotone, for short). This can be achieved by requiring all occurrences of concept names which are defined in $\mathcal{T}$ to be positive, i.e., they must be in the scope of an even number of negations [Park, 1970, Theorem 3.2]. This restriction is usually called syntactic monotonicity.

Definition 14. A terminology $\mathcal{T}$ is **syntactically monotone** iff all occurrences of concept names which are defined in $\mathcal{T}$ are **positive**, i.e., they are in the scope of an even number of negations. Clearly, $\mu\mathcal{T}$ and $\nu\mathcal{T}$ are defined to be syntactically monotone iff $\mathcal{T}$ is syntactically monotone. A complex fixpoint terminology Γ is syntactically monotone iff all the least and greatest fixpoint terminologies of Γ are syntactically monotone.

Theorem 3.2 of [Park, 1970] proves that for each syntactically monotone terminology $\mathcal{T}$ and for each interpretation $\mathcal{I}$, the function induced by $\mathcal{T}$ and $\mathcal{I}$ is monotone. Apart from the syntactic monotonicity, $\mathcal{T}$ is only restricted to be equivalent to some first-order formula. It is folklore that this is the case for all terminologies of $\mathcal{ALC}$.

Lemma 1. *Assume $\mathcal{T}$ is some syntactically monotone terminology of $\mathcal{ALC}$ comprising exactly n concept introductions and assume $\mathcal{I} = \langle \Delta^{\mathcal{I}}, .^{\mathcal{I}} \rangle$ is an arbitrary interpretation. Then the function $\mathcal{T}_{\mathcal{I}} : (2^{\Delta^{\mathcal{I}}})^n \rightarrow (2^{\Delta^{\mathcal{I}}})^n$ induced by $\mathcal{T}$ and $\mathcal{I}$ is monotone. That is to say, if $\mathcal{T}_{\mathcal{I}}(X_1^i, ..., X_n^i)$ is $\langle Y_1^i, ..., Y_n^i \rangle$, for $i = 1$ and $i = 2$, and if each X_j^1 $(1 \leq j \leq n)$ is a subset of X_j^2, then each Y_k^1 $(1 \leq k \leq n)$ is a subset of Y_k^2.*

According to the well-known Knaster-Tarski theorem, monotone functions always have least and greatest fixpoints [Tarski, 1955]. The Knaster-Tarski theorem also states that the least and greatest fixpoints of any monotone function f are unique and the least fixpoint of f is nothing but the intersection of all its fixpoints, whereas its greatest fixpoint of f the union of all its fixpoints [Tarski, 1955]. This theorem can be applied to the function $\mathcal{T}_{\mathcal{I}}$ induced by an arbitrary syntactically monotone terminology $\mathcal{T}$ of $\mathcal{ALC}$ and an arbitrary interpretation $\mathcal{I}$ because the previous lemma ensures that $\mathcal{T}_{\mathcal{I}}$ is monotone. According to Proposition 1 the result can be carried over to the corresponding least and greatest fixpoint models of $\mathcal{T}$:

Proposition 2. *Suppose $\mathcal{T}$ is some syntactically monotone terminology of $\mathcal{ALC}$. Then both $\mu\mathcal{T}$ and $\nu\mathcal{T}$ have models. Moreover, $\mathcal{I} = \langle \Delta^{\mathcal{I}}, .^{\mathcal{I}} \rangle$ is a model of $\mu\mathcal{T}$ (resp., of $\nu\mathcal{T}$) iff for all CN being defined in $\mathcal{T}$, $CN^{\mathcal{I}}$ is the intersection (resp., union) of all $CN^{\mathcal{J}}$, where $.^{\mathcal{J}}$ ranges over all models $\langle \Delta^{\mathcal{I}}, .^{\mathcal{J}} \rangle$ of $\mathcal{T}$ which are prim($\mathcal{T}$)-compatible with $\mathcal{I}$.*

Recall that a syntactically monotone complex fixpoint terminology Γ of $\mathcal{ALC}$ consists of a finite set of syntactically monotone least and greatest fixpoint terminologies of $\mathcal{ALC}$ which do not have any defined concept in common. According to the last proposition, we already know that each element of Γ does have a model. As an interpretation $\mathcal{I}$ is a model of Γ iff it is a model of each element of Γ, and as the elements of Γ do not have any defined concept in common, Γ must have a model as well. This proves the following corollary:

Corollary 1. *Each syntactically monotone complex fixpoint terminology of $\mathcal{ALC}$ has a model.*

4 The Terminological Logic $\mathcal{ALC}\mu$

So far, we dealt with least and greatest fixpoints on the metalevel rather than on the concept level. In what follows, we shall introduce an extension of $\mathcal{ALC}$, called $\mathcal{ALC}\mu$, which allows to represent least as well as greatest fixpoints explicitly by concepts of the form $\mu CN.\mathcal{T}$ and $\nu CN.\mathcal{T}$. In both expressions $\mathcal{T}$ stands for an arbitrary syntactically monotone terminology of $\mathcal{ALC}\mu$. That is to say, $\mathcal{T}$ may involve not only concepts of $\mathcal{ALC}$ but also least and greatest fixpoint operators. The meaning of these least and greatest fixpoint operators is as follows: If $\mathcal{T}$ is of the form $\{CN_i \doteq C_i : 1 \leq i \leq n\}$, then $(\mu CN_j.\mathcal{T})^{\mathcal{I}}$ and $(\nu CN_j.\mathcal{T})^{\mathcal{I}}$ represent the jth component of the least and greatest fixpoint of the function induced by $\mathcal{T}$ and $\mathcal{I}$. However, this is only the case if CN_j is actually defined in $\mathcal{T}$. If CN_j is not defined in $\mathcal{T}$, then $\mu CN_j.\mathcal{T}$ is equivalent to $\bot$, whereas $\nu CN.\mathcal{T}$ is equivalent to $\top$. We consider this extension of $\mathcal{ALC}$ because of two reasons. First of all, it is a natural extension to cope with least and greatest fixpoints explicitly and it is therefore a unifying framework for all three kinds of fixpoint semantics. Second, as a notational variant of the so-called propositional μ-calculus, $\mathcal{ALC}\mu$ itself as well as a syntactically restricted version of it are well-understood in terms of their expressive power and computational complexity. Last but not least, the syntactically restricted version of $\mathcal{ALC}\mu$ will turn out to be a suitable framework for analyzing the subsumption with respect to syntactically monotone complex fixpoint terminologies of $\mathcal{ALC}$.

Definition 15. The concepts of $\mathcal{ALC}\mu$ are inductively defined as follows:

1. Each concept name, $\bot$ and $\top$ is a concept of $\mathcal{ALC}\mu$.

2. If C and D are concepts of $\mathcal{ALC}\mu$ and if RN is a role name, then $C \sqcap D$, $C \sqcup D$, $\neg C$, $\forall RN{:}C$ and $\exists RN{:}C$ are all concepts of $\mathcal{ALC}\mu$.

3. If CN is a concept name and if $\mathcal{T}$ is some syntactically monotone terminology of $\mathcal{ALC}\mu$, then both $\mu CN.\mathcal{T}$ and $\nu CN.\mathcal{T}$ are concepts of $\mathcal{ALC}\mu$.

4. These are all concepts of $\mathcal{ALC}\mu$.

Concepts of $\mathcal{ALC}\mu$ which are of the form $\mu CN.\mathcal{T}$ or $\nu CN.\mathcal{T}$ are called **least** and **greatest fixpoint operators** respectively.

It should be stressed again that the concept introductions of $\mathcal{T}$ can be composed of *arbitrary* concepts of $\mathcal{ALC}\mu$, including least and greatest fixpoint operators, so that $\mathcal{ALC}\mu$ allows for arbitrarily nested least and greatest fixpoint operators.

We next extent the notion of an **interpretation** $\langle \Delta^{\mathcal{I}}, .^{\mathcal{I}} \rangle$ to cope with least and greatest fixpoint operators. We do so by additionally requiring that $(\mu CN.\mathcal{T})^{\mathcal{I}}$ is the intersection and $(\nu CN.\mathcal{T})^{\mathcal{I}}$ is the union of all $CN^{\mathcal{J}}$, where $.^{\mathcal{J}}$ ranges over all models $\langle \Delta^{\mathcal{I}}, .^{\mathcal{J}} \rangle$ of $\mathcal{T}$ which are $prim(\mathcal{T})$-compatible with $\mathcal{I}$. According to Proposition 2, this amounts to requiring that $\mu CN_j.\mathcal{T}$ denotes the jth component of the least fixpoint of the function induced by $\mathcal{T}$ and $\mathcal{I}$, provided that $\mathcal{T}$ is of the form $\{CN_i \doteq C_i : 1 \leq i \leq n\}$ and $1 \leq j \leq n$. Similarly, $\nu CN_j.\mathcal{T}$ denotes the jth component of the greatest fixpoint of the function $\mathcal{T}_{\mathcal{I}}$.

In [Schild, 1991], we have shown that $\mathcal{ALC}$ is nothing but a notational variant of the *propositional multi-modal logic* $\mathbf{K_{(m)}}$. The main observation is that the elements of the domain of an interpretation can be thought of as *worlds* or *states* rather than objects. Consequently, concept names can be viewed as propositional variables denoting the set of worlds in which they hold, and $\top$, $\bot$, $\sqcap$, $\sqcup$ and $\neg$ naturally correspond to the logical connectives *true*, *false*, $\wedge$, $\vee$ and to $\neg$ respectively. But then, $\forall RN$: and $\exists RN$: become RN-indexed modalities of necessity $[RN]$ and of possibility $\langle RN \rangle$. This explains why $\mathcal{ALC}$ is a notational variant of the propositional multi-modal logic $\mathbf{K_{(m)}}$. For details, the reader is referred to [Schild, 1991]. The *propositional μ-calculus with multiple fixpoints* , which has been proposed by Pratt [1981], Kozen [1983], and by Vardi and Wolper [1984], extends $\mathbf{K_{(m)}}$ by least and greatest fixpoint operators directly corresponding to those of $\mathcal{ALC}\mu$. The only difference is that $\mu CN_j.\mathcal{T}$ and $\nu CN_j.\mathcal{T}$ are written as $\mu CN_j(CN_1, ..., CN_n){:}(C_1, ..., C_n)$ and as $\nu CN_j(CN_1, ..., CN_n){:}(C_1, ..., C_n)$ if $\mathcal{T}$ is of the form $\{CN_i \doteq C_i : 1 \leq i \leq n\}$.

Correspondence Theorem 1. *$\mathcal{ALC}\mu$ is a notational variant of the propositional μ-calculus with multiple fixpoints.*

Because of this correspondence, we may assume henceforth that all results shown for the propositional μ-calculus and its variants are also shown for $\mathcal{ALC}\mu$ and the corresponding variants.

The reader may have wondered why there is only an indication of the least fixpoint operator in the name '$\mathcal{ALC}\mu$' although it extends $\mathcal{ALC}$ not only by least, but also by greatest fixpoint operators. The reason for this is that we can eliminate greatest fixpoint operators in favor of least fixpoint operators and vice versa. For instance, the least fixpoint operator $\mu A.\{A \doteq C \sqcup \exists R{:}A\}$ is equivalent to $\neg\nu A.\{A \doteq \neg(C \sqcup \exists R{:}\neg A)\}$. Notice that the terminology $\{A \doteq \neg(C \sqcup \exists R{:}\neg A)\}$ is syntactically monotone since A occurs only positively in this terminology. In fact, Park [1970] showed that this can be generalized as follows: Suppose $\widetilde{C_i}$ is obtained from C_i by replacing all occurrences of concept names defined in $\mathcal{T}$ with their negation. According to Theorem 2.3 of [Park, 1970], it then holds that:

$$\begin{aligned} &\models \neg\mu CN.\mathcal{T} \doteq \nu CN.\{CN_i \doteq \neg\widetilde{C_i} : CN_i \doteq C_i \in \mathcal{T}\} \\ &\models \neg\nu CN.\mathcal{T} \doteq \mu CN.\{CN_i \doteq \neg\widetilde{C_i} : CN_i \doteq C_i \in \mathcal{T}\} \end{aligned} \qquad (6)$$

As $\neg\widetilde{C_i}$ adds exactly two negations to the scope of all occurrence of concept names which are defined in $\mathcal{T}$, the terminology $\{CN_i \doteq \neg\widetilde{C_i} : CN_i \doteq C_i \in \mathcal{T}\}$ is clearly syntactically monotone iff $\mathcal{T}$ is syntactically monotone. These equivalences are crucial to obtain the negation normal form of concepts of $\mathcal{ALC}\mu$. As usual, the negation normal form of a concept is an equivalent concept in which no compound concept is negated. For concepts of $\mathcal{ALC}$, it can be obtained by exploiting de Morgan's laws as well as the equivalences

$\models \neg\forall R{:}C \doteq \exists R{:}\neg C$ and $\models \neg\exists R{:}C \doteq \forall R{:}\neg C$. In case of $\mathcal{ALC}\mu$, however, we have to exploit additionally the two equivalences above (6).

Definition 16. The function ***nf*** maps concepts of $\mathcal{ALC}\mu$ to concepts of $\mathcal{ALC}\mu$ and is inductively defined as follows:

1. If $C = C_1 \sqcap C_2$, then $nf(C)$ is $nf(C_1) \sqcap nf(C_2)$.
2. If $C = C_1 \sqcup C_2$, then $nf(C)$ is $nf(C_1) \sqcup nf(C_2)$.
3. If $C = \forall R{:}D$, then $nf(C)$ is $\forall R{:}nf(D)$.
4. If $C = \exists R{:}D$, then $nf(C)$ is $\exists R{:}nf(D)$.
5. If $C = \mu CN.\mathcal{T}$, then $nf(C)$ is $\mu CN.\{CN_i \doteq nf(C_i) : CN_i \doteq C_i \in \mathcal{T}\}$.
6. If $C = \nu CN.\mathcal{T}$, then $nf(C)$ is $\nu CN.\{CN_i \doteq nf(C_i) : CN_i \doteq C_i \in \mathcal{T}\}$.
7. If $C = \neg(C_1 \sqcap C_2)$, then $nf(C)$ is $nf(\neg C_1) \sqcup nf(\neg C_2)$.
8. If $C = \neg(C_1 \sqcup C_2)$, then $nf(C)$ is $nf(\neg C_1) \sqcap nf(\neg C_2)$.
9. If $C = \neg\forall R{:}D$, then $nf(C)$ is $\exists R{:}nf(\neg D)$.
10. If $C = \neg\exists R{:}D$, then $nf(C)$ is $\forall R{:}nf(\neg D)$.
11. If $C = \neg\mu CN.\mathcal{T}$, then $nf(C)$ is $\nu CN.\{CN_i \doteq nf(\neg\widetilde{C_i}) : CN_i \doteq C_i \in \mathcal{T}\}$.
12. If $C = \neg\nu CN.\mathcal{T}$, then $nf(C)$ is $\mu CN.\{CN_i \doteq nf(\neg\widetilde{C_i}) : CN_i \doteq C_i \in \mathcal{T}\}$.
13. Otherwise $nf(C)$ is C.

As above $\widetilde{C_i}$ is obtained from C_i by replacing all occurrences of concept names which are defined in $\mathcal{T}$ with their negations. We call $nf(C)$ to be the **negation normal form** of C.

Lemma 2. *Every concept of $\mathcal{ALC}\mu$ is equivalent to its negation normal form.*

Its worth mentioning that the meaning of the concepts $\mu CN.\mathcal{T}$ and $\nu CN.\mathcal{T}$ is preserved by renaming each concept name which is defined in $\mathcal{T}$, so that the concept names which are defined in $\mathcal{T}$ behave in $\mu CN.\mathcal{T}$ and in $\nu CN.\mathcal{T}$ like quantified variables. The following lemma due to Kozen [1983, Proposition 5.7(i)] is devoted to this renaming.

Lemma 3. *Assume $\mathcal{T}$ is an arbitrary terminology in which CN_i is defined but which does not contain an occurrence of the concept name A_i. Suppose, moreover, $(\mu CN_j.\mathcal{T})_{CN_i/A_i}$ and $(\nu CN_j.\mathcal{T})_{CN_i/A_i}$ are obtained from $\mu CN_j.\mathcal{T}$ and from $\nu CN_j.\mathcal{T}$ by replacing each occurrence of CN_i with A_i. Then the following equivalences hold, even if i and j are not distinct:*

$$\models \mu CN_j.\mathcal{T} \doteq (\mu CN_j.\mathcal{T})_{CN_i/A_i}$$
$$\models \nu CN_j.\mathcal{T} \doteq (\nu CN_j.\mathcal{T})_{CN_i/A_i}$$

Although coherence in full $\mathcal{ALC}\mu$ is known to be elementarily decidable [Streett and Emerson, 1984, Streett and Emerson, 1989], only many-fold exponential upper bounds are known. To gain an one exponential time upper bound, the interaction of nested *alternating* least and greatest fixpoints must be limited [Vardi and Wolper, 1984]. Loosely speaking, the definition of restrictedness below requires that nested *alternating* least and greatest fixpoints may not interact via a defined concept. However, restricted concepts of $\mathcal{ALC}\mu$ will suffice to represent syntactically monotone complex fixpoint terminology.

Definition 17. A concept C of $\mathcal{ALC}\mu$ is called **restricted** iff its negation normal form does not contain any least fixpoint operator $\mu CN.\mathcal{T}$ which involves some greatest fixpoint operator in which a concept defined in $\mathcal{T}$ occurs and, moreover, its negation normal form does not contain any greatest fixpoint operator $\nu CN.\mathcal{T}$ which involves some least fixpoint operator in which a concept defined in $\mathcal{T}$ occurs. We denote with $\boldsymbol{\mathcal{ALC}\mu^-}$ the set of all restricted concepts of $\mathcal{ALC}\mu$.

Consider, for instance, the concept $\mu\boldsymbol{A}.\{\boldsymbol{A} \doteq \forall R{:}\nu B.\{B \doteq \boldsymbol{A} \sqcap \forall S{:}B\}\}$. First of all, it is already in negation normal form. Second, it comprises a greatest fixpoint operator, viz. $\nu B.\{B \doteq \boldsymbol{A} \sqcap \forall S{:}B\}$, which is nested in a least fixpoint operator of the form $\mu\boldsymbol{A}.\mathcal{T}$. As there is an occurrence of $\boldsymbol{A}$ in the greatest fixpoint operator $\nu B.\{B \doteq \boldsymbol{A} \sqcap \forall S{:}B\}$, the concept above is not restricted and is therefore no concept of $\mathcal{ALC}\mu^-$. Observe, however, that $\mu A.\{A \doteq \forall R{:}\mu B.\{B \doteq A \sqcap \forall S{:}B\}\}$ *is* restricted.

In view of the fact that many μ-calculi considered in the literature does not allow for mutual fixpoints, we should clarify the actual role of multiple fixpoints. By *mutual fixpoints*, we mean least or greatest fixpoint operators applied to terminologies comprising more than one concept introduction. It turns out that we can eliminate mutual fixpoints in favor of nested ones. Consider, for instance, the mutual fixpoint $\nu A.\{A \doteq \forall R{:}B, B \doteq \forall S{:}(A \sqcap B)\}$. This concept is in fact equivalent to $\nu A.\{A \doteq \forall R{:}\nu B.\{B \doteq \forall S{:}(A \sqcap B)\}\}$, which obviously does not contain any mutual fixpoint. Now, take the general case, viz. $\nu CN_j.\mathcal{T}$ with $\mathcal{T}$ being $\{CN_i \doteq C_i : 1 \leq i \leq n\}$. As a rule, we may eliminate each concept introduction $CN_l \doteq C_l$ of $\mathcal{T}$ with $l \neq j$ if all occurrences of CN_l are simultaneously replaced with $\nu CN_l.\{CN_l \doteq C_l\}$.

Proposition 3. *Assume $\mathcal{T}$ is a syntactically monotone terminology of $\mathcal{ALC}\mu$ which is of the form $\{CN_i \doteq C_i : 1 \leq i \leq n\}$. Assume without loss of generality that there is no occurrence of the form $\mu CN.\mathcal{T}'$ or $\nu CN.\mathcal{T}'$ in $C_1, ..., C_n$ if CN is defined in $\mathcal{T}$. Finally, assume $\hat{C}_j$ and $\check{C}_j$ are obtained from C_j by simultaneously replacing each occurrence of CN_l with $\mu CN_l.\{CN_l \doteq C_l\}$ and with $\nu CN_l.\{CN_l \doteq C_l\}$ respectively. If $j \neq l$, then $\mu CN_j.\mathcal{T}$ is equivalent to $\mu CN_j.\{CN_j \doteq \hat{C}_j\}$ and $\nu CN_j.\mathcal{T}$ is equivalent to $\nu CN_j.\{CN_j \doteq \check{C}_j\}$.*

A proof of this proposition is given by, e.g., De Bakker [1980, Theorem 5.14.e]. Of course, a finite number of applications of Proposition 3 eliminates all mutual fixpoints. Remarkably, this works also for concepts of $\mathcal{ALC}\mu^-$ as this language restricts only the interaction of nested *alternating* least and greatest fixpoints operators which are not used in Proposition 3.

Corollary 2. *Every concept of $\mathcal{ALC}\mu$ is equivalent to a concept of $\mathcal{ALC}\mu$ which involves solely terminologies which contain at most one concept introduction. The corresponding statement holds for $\mathcal{ALC}\mu^-$ as well.*

Unfortunately, the size of equivalent concept is not always bounded polynomially in the size of the original concept.

5 Representing Syntactically Monotone Fixpoint Terminologies of $\mathcal{ALC}$ as Terminologies of $\mathcal{ALC}\mu^-$

As Proposition 2 suggests, there is an intimate relationship between syntactically monotone least and greatest fixpoint terminologies on the one hand and least and greatest fixpoint operators of $\mathcal{ALC}\mu$ on the other hand. The least fixpoint terminology $\mu\{CN \doteq C\}$, for instance, has exactly the same models as the concept introduction $CN \doteq \mu A.\{A \doteq C_{CN/A}\}$ if A is a concept name not occurring in C. As usual, $C_{CN/A}$ is obtained from C by replacing all occurrences of CN with A. Note, however, the concept names which are defined in $\mathcal{T}$ behave in $\mu A.\mathcal{T}$ and in $\nu A.\mathcal{T}$ like quantified variables in that they have *local* meaning, whereas those defined in fixpoint terminologies have *global* meaning. This is due to the fact that according to Lemma 3, the meaning of the concepts $\mu A.\mathcal{T}$ and $\nu A.\mathcal{T}$ is in fact preserved by *renaming* each concept name which is defined in $\mathcal{T}$. In contrast to this, renaming a concept which is defined in $\mathcal{T}$ does change the meaning of the least and the greatest fixpoint terminology $\mu\mathcal{T}$ and $\nu\mathcal{T}$. Therefore, for *each* concept introduction $CN_i \doteq C_i$ of the least fixpoint terminology $\mu\mathcal{T}$, a concept introduction $CN_i \doteq \mu A_i.\mathcal{T}_{CN_i/A_i}$ is needed. In this way, every model $\mathcal{I}$ of these concept introductions forces each CN_i which is defined in $\mathcal{T}$ to be the ith

Lemma 3. *Assume $\mathcal{T}$ is an arbitrary terminology in which CN_i is defined but which does not contain an occurrence of the concept name A_i. Suppose, moreover, $(\mu CN_j.\mathcal{T})_{CN_i/A_i}$ and $(\nu CN_j.\mathcal{T})_{CN_i/A_i}$ are obtained from $\mu CN_j.\mathcal{T}$ and from $\nu CN_j.\mathcal{T}$ by replacing each occurrence of CN_i with A_i. Then the following equivalences hold, even if i and j are not distinct:*

$$\models \mu CN_j.\mathcal{T} \doteq (\mu CN_j.\mathcal{T})_{CN_i/A_i}$$
$$\models \nu CN_j.\mathcal{T} \doteq (\nu CN_j.\mathcal{T})_{CN_i/A_i}$$

Although coherence in full $\mathcal{ALC}\mu$ is known to be elementarily decidable [Streett and Emerson, 1984, Streett and Emerson, 1989], only many-fold exponential upper bounds are known. To gain an one exponential time upper bound, the interaction of nested *alternating* least and greatest fixpoints must be limited [Vardi and Wolper, 1984]. Loosely speaking, the definition of restrictedness below requires that nested *alternating* least and greatest fixpoints may not interact via a defined concept. However, restricted concepts of $\mathcal{ALC}\mu$ will suffice to represent syntactically monotone complex fixpoint terminology.

Definition 17. A concept C of $\mathcal{ALC}\mu$ is called **restricted** iff its negation normal form does not contain any least fixpoint operator $\mu CN.\mathcal{T}$ which involves some greatest fixpoint operator in which a concept defined in $\mathcal{T}$ occurs and, moreover, its negation normal form does not contain any greatest fixpoint operator $\nu CN.\mathcal{T}$ which involves some least fixpoint operator in which a concept defined in $\mathcal{T}$ occurs. We denote with $\boldsymbol{\mathcal{ALC}\mu^-}$ the set of all restricted concepts of $\mathcal{ALC}\mu$.

Consider, for instance, the concept $\mu\boldsymbol{A}.\{\boldsymbol{A} \doteq \forall R{:}\nu B.\{B \doteq \boldsymbol{A} \sqcap \forall S{:}B\}\}$. First of all, it is already in negation normal form. Second, it comprises a greatest fixpoint operator, viz. $\nu B.\{B \doteq \boldsymbol{A} \sqcap \forall S{:}B\}$, which is nested in a least fixpoint operator of the form $\mu\boldsymbol{A}.\mathcal{T}$. As there is an occurrence of $\boldsymbol{A}$ in the greatest fixpoint operator $\nu B.\{B \doteq \boldsymbol{A} \sqcap \forall S{:}B\}$, the concept above is not restricted and is therefore no concept of $\mathcal{ALC}\mu^-$. Observe, however, that $\mu A.\{A \doteq \forall R{:}\mu B.\{B \doteq A \sqcap \forall S{:}B\}\}$ *is* restricted.

In view of the fact that many μ-calculi considered in the literature does not allow for mutual fixpoints, we should clarify the actual role of multiple fixpoints. By *mutual fixpoints*, we mean least or greatest fixpoint operators applied to terminologies comprising more than one concept introduction. It turns out that we can eliminate mutual fixpoints in favor of nested ones. Consider, for instance, the mutual fixpoint $\nu A.\{A \doteq \forall R{:}B, B \doteq \forall S{:}(A \sqcap B)\}$. This concept is in fact equivalent to $\nu A.\{A \doteq \forall R{:}\nu B.\{B \doteq \forall S{:}(A \sqcap B)\}\}$, which obviously does not contain any mutual fixpoint. Now, take the general case, viz. $\nu CN_j.\mathcal{T}$ with $\mathcal{T}$ being $\{CN_i \doteq C_i : 1 \leq i \leq n\}$. As a rule, we may eliminate each concept introduction $CN_l \doteq C_l$ of $\mathcal{T}$ with $l \neq j$ if all occurrences of CN_l are simultaneously replaced with $\nu CN_l.\{CN_l \doteq C_l\}$.

Proposition 3. *Assume $\mathcal{T}$ is a syntactically monotone terminology of $\mathcal{ALC}\mu$ which is of the form $\{CN_i \doteq C_i : 1 \leq i \leq n\}$. Assume without loss of generality that there is no occurrence of the form $\mu CN.\mathcal{T}'$ or $\nu CN.\mathcal{T}'$ in $C_1, ..., C_n$ if CN is defined in $\mathcal{T}$. Finally, assume $\hat{C}_j$ and $\check{C}_j$ are obtained from C_j by simultaneously replacing each occurrence of CN_l with $\mu CN_l.\{CN_l \doteq C_l\}$ and with $\nu CN_l.\{CN_l \doteq C_l\}$ respectively. If $j \neq l$, then $\mu CN_j.\mathcal{T}$ is equivalent to $\mu CN_j.\{CN_j \doteq \hat{C}_j\}$ and $\nu CN_j.\mathcal{T}$ is equivalent to $\nu CN_j.\{CN_j \doteq \check{C}_j\}$.*

A proof of this proposition is given by, e.g., De Bakker [1980, Theorem 5.14.e]. Of course, a finite number of applications of Proposition 3 eliminates all mutual fixpoints. Remarkably, this works also for concepts of $\mathcal{ALC}\mu^-$ as this language restricts only the interaction of nested *alternating* least and greatest fixpoints operators which are not used in Proposition 3.

Corollary 2. *Every concept of $\mathcal{ALC}\mu$ is equivalent to a concept of $\mathcal{ALC}\mu$ which involves solely terminologies which contain at most one concept introduction. The corresponding statement holds for $\mathcal{ALC}\mu^-$ as well.*

Unfortunately, the size of equivalent concept is not always bounded polynomially in the size of the original concept.

5 Representing Syntactically Monotone Fixpoint Terminologies of $\mathcal{ALC}$ as Terminologies of $\mathcal{ALC}\mu^-$

As Proposition 2 suggests, there is an intimate relationship between syntactically monotone least and greatest fixpoint terminologies on the one hand and least and greatest fixpoint operators of $\mathcal{ALC}\mu$ on the other hand. The least fixpoint terminology $\mu\{CN \doteq C\}$, for instance, has exactly the same models as the concept introduction $CN \doteq \mu A.\{A \doteq C_{CN/A}\}$ if A is a concept name not occurring in C. As usual, $C_{CN/A}$ is obtained from C by replacing all occurrences of CN with A. Note, however, the concept names which are defined in $\mathcal{T}$ behave in $\mu A.\mathcal{T}$ and in $\nu A.\mathcal{T}$ like quantified variables in that they have *local* meaning, whereas those defined in fixpoint terminologies have *global* meaning. This is due to the fact that according to Lemma 3, the meaning of the concepts $\mu A.\mathcal{T}$ and $\nu A.\mathcal{T}$ is in fact preserved by *renaming* each concept name which is defined in $\mathcal{T}$. In contrast to this, renaming a concept which is defined in $\mathcal{T}$ does change the meaning of the least and the greatest fixpoint terminology $\mu\mathcal{T}$ and $\nu\mathcal{T}$. Therefore, for *each* concept introduction $CN_i \doteq C_i$ of the least fixpoint terminology $\mu\mathcal{T}$, a concept introduction $CN_i \doteq \mu A_i.\mathcal{T}_{CN_i/A_i}$ is needed. In this way, every model $\mathcal{I}$ of these concept introductions forces each CN_i which is defined in $\mathcal{T}$ to be the ith component of the function induced by $\mathcal{T}$ and $\mathcal{I}$, so that $\mathcal{I}$ is in fact a model of $\mu\mathcal{T}$.

Proposition 4. *Assume $\mathcal{T}$ is some syntactically monotone terminology of $\mathcal{ALC}$ which is of the form $\{CN_i \doteq C_i : 1 \leq i \leq n\}$ and which does not contain any of the pairwise distinct concept names $A_1, \ldots, A_n$. Assume furthermore that $\mathcal{T}_A$ is obtained from $\mathcal{T}$ by replacing for all i with $1 \leq i \leq n$ each occurrence of CN_i with A_i. It then holds that*

1. *$\mu\mathcal{T}$ has the same models as $\{CN_i \doteq \mu A_i.\mathcal{T}_A : 1 \leq i \leq n\}$ and*

2. *$\nu\mathcal{T}$ has the same models as $\{CN_i \doteq \nu A_i.\mathcal{T}_A : 1 \leq i \leq n\}$.*

This proposition describes how to represent least and greatest fixpoint terminologies of $\mathcal{ALC}$ as terminologies of $\mathcal{ALC}\mu$. It should be remarked that we could have taken also $\mu CN_i.\mathcal{T}$ and $\nu CN_i.\mathcal{T}$ instead of $\mu A_i.\mathcal{T}_A$ and $\nu A_i.\mathcal{T}_A$ because according to Lemma 3 they are equivalent. However, we have taken the terminologies $\{CN_i \doteq \mu A_i.\mathcal{T}_A : 1 \leq i \leq n\}$ and $\{CN_i \doteq \nu A_i.\mathcal{T}_A : 1 \leq i \leq n\}$ because they are acyclic. Let us take a closer look at these terminologies. $\mathcal{T}_A$ is clearly a terminology of $\mathcal{ALC}$ since $\mathcal{T}$ is assumed to be a terminology of $\mathcal{ALC}$. Therefore, both $\mu A_i.\mathcal{T}_A$ and $\nu A_i.\mathcal{T}_A$ are restricted because they do not contain any nested fixpoint operator. This means, the last proposition can be interpreted as stating that every syntactically monotone least and greatest fixpoint terminology can be represented as an acyclic terminology of $\mathcal{ALC}\mu^-$. The following theorem states that syntactically monotone *complex* fixpoint terminologies of $\mathcal{ALC}$ can be represented in this way as well.

Representation Theorem 1. *There is a function π which maps an arbitrary syntactically monotone complex fixpoint terminology Γ of $\mathcal{ALC}$ to some acyclic terminology $\pi(\Gamma)$ of $\mathcal{ALC}\mu^-$ such that Γ and $\pi(\Gamma)$ have exactly the same models. Additionally, π is computable in polynomial time and the size of $\pi(\Gamma)$ is linearly bounded in the size of Γ.*

By **size**, we mean the length when considered as a string over $\mathcal{N}$, $\sqcap$, $\sqcup$, $\neg$, $\forall$, $\exists$, μ and ν.

Proof. Assume $\mathcal{T}$ and $\mathcal{T}_A$ are given as in Proposition 4. According to Proposition 4, we already know that $\Gamma \cup \{\mu\mathcal{T}\}$ and $\Gamma \cup \{CN_i \doteq \mu A_i.\mathcal{T}_A : 1 \leq i \leq n\}$ do have the same models and that the analogous statement holds for $\nu\mathcal{T}$ as well. As $\mathcal{T}_A$ is a terminology of $\mathcal{ALC}$, it does not contain any least or greatest fixpoint operator, so that the concepts $\mu A_i.\mathcal{T}_A$ and $\nu A_i.\mathcal{T}_A$ are always restricted. Eliminating one least or greatest fixpoint terminology in this way is clearly computable in polynomial time and the size of the resulting set is linearly bounded in the size of the original one, so that induction on the number of least and greatest fixpoint terminologies proves the assertion. □

6 Expressive Power

In this section, we shall see that the concepts definable by syntactically monotone complex fixpoint terminologies of $\mathcal{ALC}$ are are exactly the concepts equivalent to those of $\mathcal{ALC}\mu^-$. This holds even if the complex fixpoint terminologies are restricted to contain solely least (resp., greatest) fixpoint terminologies. We then give a strict lower bound of the expressive power of $\mathcal{ALC}\mu^-$ and of full $\mathcal{ALC}\mu$ in terms of $\mathcal{ALC}$ augmented by regular and ω-regular role expressions. Of course, before engaging into details,we have to clarify what we mean by expressive power and definability.

Definition 18. Suppose $\mathcal{L}$ and $\mathcal{L}'$ are two sets of concepts. Then $\mathcal{L}$ is **at least as strong in expressive power** as $\mathcal{L}'$, $\mathcal{L}' \leq \mathcal{L}$ for short, iff for each concept in $\mathcal{L}'$ there is an equivalent one in $\mathcal{L}$, and $\mathcal{L}$ is **strictly stronger in expressive power** than $\mathcal{L}'$ iff $\mathcal{L}' \leq \mathcal{L}$ but it is not the case that $\mathcal{L} \leq \mathcal{L}'$. Furthermore, a concept C is **definable** by a set of complex fixpoint terminologies of $\mathcal{L}$ iff there is an element Γ of the set and there is a concept name CN which is defined in Γ such that $\Gamma \models CN \doteq C$.

That is to say, $\mathcal{L}$ is at least as strong in expressive power as $\mathcal{L}'$ just in case that for each concept in $\mathcal{L}'$ there is a concept in $\mathcal{L}$ which has exactly the same meaning, though the two concepts may differ in their syntax. If additionally there is a concept in $\mathcal{L}$ which is not equivalent to any concept of $\mathcal{L}'$, $\mathcal{L}$ is said to be strictly stronger in expressive power than $\mathcal{L}'$. For example, it can be shown that $\mathcal{ALC}$ augmented by the reflexive-transitive closure of roles is strictly stronger in expressive power than $\mathcal{ALC}$. The definition of definability of concepts takes into account the fact that the definitional power of terminologies consists in their defined concept names. The concept $\exists RN^*{:}C$, for instance, is definable by syntactically monotone complex fixpoint terminologies of $\mathcal{ALC}$ since $\mu\{A \doteq C \sqcup \exists RN{:}A\} \models A \doteq \exists RN^*{:}C$.

Expressiveness Theorem 1. *The concepts definable by syntactically monotone complex fixpoint terminologies of $\mathcal{ALC}$ are exactly the concepts equivalent to those of $\mathcal{ALC}\mu^-$.*

Proof. First, we show that each concept C of $\mathcal{ALC}\mu^-$ is definable by syntactically monotone complex terminologies of $\mathcal{ALC}$. According to Lemma 2, we may assume that C is in negation normal form and, additionally, according to Lemma 3, we may assume that there are no two occurrences of terminologies $\mathcal{T}$ and $\mathcal{T}'$ in C which have at least one defined concept in common. Now, although C is restricted, C may contain nested fixpoint operators whose interaction is not limited. It may contain, for instance, an occurrence of $\mu \boldsymbol{A}.\{\boldsymbol{A} \doteq \forall R{:}\mu B.\{B \doteq \boldsymbol{A} \sqcap \forall S{:}B\}\}$. In such a case, Proposition 3 has to be applied. Proposition 3 states that this concept is equivalent component of the function induced by $\mathcal{T}$ and $\mathcal{I}$, so that $\mathcal{I}$ is in fact a model of $\mu\mathcal{T}$.

Proposition 4. *Assume $\mathcal{T}$ is some syntactically monotone terminology of $\mathcal{ALC}$ which is of the form $\{CN_i \doteq C_i : 1 \leq i \leq n\}$ and which does not contain any of the pairwise distinct concept names $A_1, ..., A_n$. Assume furthermore that $\mathcal{T}_A$ is obtained from $\mathcal{T}$ by replacing for all i with $1 \leq i \leq n$ each occurrence of CN_i with A_i. It then holds that*

1. *$\mu\mathcal{T}$ has the same models as $\{CN_i \doteq \mu A_i.\mathcal{T}_A : 1 \leq i \leq n\}$ and*

2. *$\nu\mathcal{T}$ has the same models as $\{CN_i \doteq \nu A_i.\mathcal{T}_A : 1 \leq i \leq n\}$.*

This proposition describes how to represent least and greatest fixpoint terminologies of $\mathcal{ALC}$ as terminologies of $\mathcal{ALC}\mu$. It should be remarked that we could have taken also $\mu CN_i.\mathcal{T}$ and $\nu CN_i.\mathcal{T}$ instead of $\mu A_i.\mathcal{T}_A$ and $\nu A_i.\mathcal{T}_A$ because according to Lemma 3 they are equivalent. However, we have taken the terminologies $\{CN_i \doteq \mu A_i.\mathcal{T}_A : 1 \leq i \leq n\}$ and $\{CN_i \doteq \nu A_i.\mathcal{T}_A : 1 \leq i \leq n\}$ because they are acyclic. Let us take a closer look at these terminologies. $\mathcal{T}_A$ is clearly a terminology of $\mathcal{ALC}$ since $\mathcal{T}$ is assumed to be a terminology of $\mathcal{ALC}$. Therefore, both $\mu A_i.\mathcal{T}_A$ and $\nu A_i.\mathcal{T}_A$ are restricted because they do not contain any nested fixpoint operator. This means, the last proposition can be interpreted as stating that every syntactically monotone least and greatest fixpoint terminology can be represented as an acyclic terminology of $\mathcal{ALC}\mu^-$. The following theorem states that syntactically monotone *complex* fixpoint terminologies of $\mathcal{ALC}$ can be represented in this way as well.

Representation Theorem 1. *There is a function π which maps an arbitrary syntactically monotone complex fixpoint terminology Γ of $\mathcal{ALC}$ to some acyclic terminology $\pi(\Gamma)$ of $\mathcal{ALC}\mu^-$ such that Γ and $\pi(\Gamma)$ have exactly the same models. Additionally, π is computable in polynomial time and the size of $\pi(\Gamma)$ is linearly bounded in the size of Γ.*

By **size**, we mean the length when considered as a string over $\mathcal{N}$, $\sqcap$, $\sqcup$, $\neg$, $\forall$, $\exists$, μ and ν.

Proof. Assume $\mathcal{T}$ and $\mathcal{T}_A$ are given as in Proposition 4. According to Proposition 4, we already know that $\Gamma \cup \{\mu\mathcal{T}\}$ and $\Gamma \cup \{CN_i \doteq \mu A_i.\mathcal{T}_A : 1 \leq i \leq n\}$ do have the same models and that the analogous statement holds for $\nu\mathcal{T}$ as well. As $\mathcal{T}_A$ is a terminology of $\mathcal{ALC}$, it does not contain any least or greatest fixpoint operator, so that the concepts $\mu A_i.\mathcal{T}_A$ and $\nu A_i.\mathcal{T}_A$ are always restricted. Eliminating one least or greatest fixpoint terminology in this way is clearly computable in polynomial time and the size of the resulting set is linearly bounded in the size of the original one, so that induction on the number of least and greatest fixpoint terminologies proves the assertion. □

6 Expressive Power

In this section, we shall see that the concepts definable by syntactically monotone complex fixpoint terminologies of $\mathcal{ALC}$ are are exactly the concepts equivalent to those of $\mathcal{ALC}\mu^-$. This holds even if the complex fixpoint terminologies are restricted to contain solely least (resp., greatest) fixpoint terminologies. We then give a strict lower bound of the expressive power of $\mathcal{ALC}\mu^-$ and of full $\mathcal{ALC}\mu$ in terms of $\mathcal{ALC}$ augmented by regular and ω-regular role expressions. Of course, before engaging into details,we have to clarify what we mean by expressive power and definability.

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That is to say, $\mathcal{L}$ is at least as strong in expressive power as $\mathcal{L}'$ just in case that for each concept in $\mathcal{L}'$ there is a concept in $\mathcal{L}$ which has exactly the same meaning, though the two concepts may differ in their syntax. If additionally there is a concept in $\mathcal{L}$ which is not equivalent to any concept of $\mathcal{L}'$, $\mathcal{L}$ is said to be strictly stronger in expressive power than $\mathcal{L}'$. For example, it can be shown that $\mathcal{ALC}$ augmented by the reflexive-transitive closure of roles is strictly stronger in expressive power than $\mathcal{ALC}$. The definition of definability of concepts takes into account the fact that the definitional power of terminologies consists in their defined concept names. The concept $\exists RN^*{:}C$, for instance, is definable by syntactically monotone complex fixpoint terminologies of $\mathcal{ALC}$ since $\mu\{A \doteq C \sqcup \exists RN{:}A\} \models A \doteq \exists RN^*{:}C$.

Expressiveness Theorem 1. *The concepts definable by syntactically monotone complex fixpoint terminologies of $\mathcal{ALC}$ are exactly the concepts equivalent to those of $\mathcal{ALC}\mu^-$.*

Proof. First, we show that each concept C of $\mathcal{ALC}\mu^-$ is definable by syntactically monotone complex terminologies of $\mathcal{ALC}$. According to Lemma 2, we may assume that C is in negation normal form and, additionally, according to Lemma 3, we may assume that there are no two occurrences of terminologies $\mathcal{T}$ and $\mathcal{T}'$ in C which have at least one defined concept in common. Now, although C is restricted, C may contain nested fixpoint operators whose interaction is not limited. It may contain, for instance, an occurrence of $\mu\boldsymbol{A}.\{\boldsymbol{A} \doteq \forall R{:}\mu B.\{B \doteq \boldsymbol{A} \sqcap \forall S{:}B\}\}$. In such a case, Proposition 3 has to be applied. Proposition 3 states that this concept is equivalent to $\mu A.\{A \doteq \forall R{:}B, B \doteq A \sqcap \forall S{:}B\}$. That means, we may even assume that there is no occurrence of $\mu CN.\mathcal{T}$ or $\nu CN.\mathcal{T}$ in C such that $\mathcal{T}$ contains a least or greatest fixpoint operator in which some concept defined in $\mathcal{T}$ occurs. Now, take an occurrence of a the form $\mu CN_i.\mathcal{T}$ or $\nu CN_i.\mathcal{T}$ in C such that $\mathcal{T}$ contains only concepts of $\mathcal{ALC}$. The concept name CN_i can be assumed to be defined in $\mathcal{T}$ since otherwise $\mu CN_i.\mathcal{T}$ and $\nu CN_i.\mathcal{T}$ would be equivalent to $\bot$ and $\top$ respectively. We consider only the case $\mu CN_i.\mathcal{T}$ since the other one can be shown in the very analogous way. Proposition 4 tells us that $\mu\mathcal{T} \models CN_i \doteq \mu A_i.\mathcal{T}_A$, where $\mathcal{T}_A$ is defined exactly as in Proposition 4. Recall that each concept name which is defined in $\mathcal{T}_A$ can be renamed without changing the meaning of $\mu A_i.\mathcal{T}_A$. Therefore, $\mu A_i.\mathcal{T}_A$ is in fact equivalent to $\mu CN_i.\mathcal{T}$, so that $\mu\mathcal{T} \models CN_i \doteq \mu CN_i.\mathcal{T}$. According to the previously made assumptions $\mu CN_i.\mathcal{T}$ does not contain any concept name which is 'bounded' by some other least and greatest fixpoint operator in C. That is, $\mu CN_i.\mathcal{T}$ contains no concept name such that there is an other occurrence of the form $\mu CN'.\mathcal{T}'$ or $\nu CN'.\mathcal{T}'$ in C such that CN_i is defined in $\mathcal{T}'$. This ensures that $\mu\mathcal{T} \models C_{\mu CN_i.\mathcal{T}/CN_i} \doteq C$ if $C_{\mu CN_i.\mathcal{T}/CN_i}$ is obtained from C by simultaneously replacing all occurrences of $\mu CN_i.\mathcal{T}$ with CN_i. Clearly, $C_{\mu CN_i.\mathcal{T}/CN_i}$ comprises $n-1$ occurrences of least fixpoint operators if C has n occurrences. Induction on n then shows there is syntactically monotone complex fixpoint terminology Γ of $\mathcal{ALC}$ such that $\Gamma \models D \doteq C$, where D is some concept of $\mathcal{ALC}$. This clearly holds iff $\Gamma \cup \{\mu\{A \doteq D\}\} \models A \doteq C$, provided A is a fresh concept name not occurring in C, D and Γ. The obvious fact that $\{A \doteq D\}$ is syntactically monotone immediately shows C to be in fact definable by syntactically monotone complex fixpoint terminologies of $\mathcal{ALC}$.

It remains to prove that if C is not definable by any syntactically monotone complex fixpoint terminology of $\mathcal{ALC}$, then C is not equivalent to any concept of $\mathcal{ALC}\mu^-$. The proof proceeds by *reductio ad absurdum*, i.e., asserting the contrary will be shown to yield a contradiction. So, suppose there is a syntactically monotone complex fixpoint terminology Γ of $\mathcal{ALC}$ and there exists some concept C not equivalent to any concept of $\mathcal{ALC}\mu^-$ such that $\Gamma \models CN \doteq C$. According to Representation Theorem 1 there is an acyclic terminology $\mathcal{T}$ of $\mathcal{ALC}\mu^-$ such that $\mathcal{T} \models CN \doteq C$. As $\mathcal{T}$ is an acyclic terminology, we can abstract from it in the usual way. This can be achieved by replacing CN with D if $CN \doteq D \in \mathcal{T}$ and then repeatedly replacing in D each CN' such that $CN' \doteq D' \in \mathcal{T}$ with D' [Nebel, 1990, Chapter 3.2.5]. That means, there is a concept C' of $\mathcal{ALC}\mu^-$ which is equivalent to C. However, this contradicts the assumption that C is not equivalent to any concept of $\mathcal{ALC}\mu^-$. □

The equivalences (6) imply together with Proposition 4 that for every syntactically monotone complex fixpoint terminology of $\mathcal{ALC}$ there is one which contains solely least (resp., greatest) fixpoint terminologies. .To see this consider an arbitrary syntactically monotone complex fixpoint terminology Γ of $\mathcal{ALC}$ which contains $\nu\mathcal{T}$. Assume $\mathcal{T}$ is of the form $\{CN_i \doteq C_i : 1 \leq i \leq n\}$ and assume it does not contain the concept names $A_1, ..., A_n$ which are pairwise distinct. Assume, moreover, $\mathcal{T}_A$ is obtained from $\mathcal{T}$ by replacing for every i with $1 \leq i \leq n$, each occurrence occurrence of CN_i with A_i. Then $\nu\mathcal{T}$ does have the same models as the following complex fixpoint terminology of $\mathcal{ALC}$:

$$\mu\{A_i \doteq \neg\widetilde{D_i} : A_i \doteq D_i \in \mathcal{T}_A\} \;\cup\; \mu\{CN_i \doteq \neg A_i : 1 \leq i \leq n\}$$

As above $\widetilde{D_i}$ is obtained from D_i by replacing for every i $(1 \leq i \leq n)$ each occurrence of A_i with $\neg A_i$. It should be clear that this complex fixpoint terminology is syntactically monotone since so is $\mathcal{T}$. This proves that each greatest fixpoint terminology of Γ can be replaced with two least fixpoint terminologies without changing the models of Γ. Of course, all least fixpoint terminologies of Γ can be replaced with two greatest fixpoint terminologies similarly. This proves the following theorem:

Expressiveness Theorem 2. *The concepts definable by syntactically monotone complex fixpoint terminologies of $\mathcal{ALC}$ are exactly the concepts definable by syntactically monotone complex fixpoint terminologies of $\mathcal{ALC}$ which contain solely least (resp., greatest) fixpoint terminologies.*

Now, recall that Corollary 2 states that each concept of $\mathcal{ALC}\mu$ is equivalent to one which involves solely terminologies comprising at most one concept introduction and that this holds also for restricted concepts. The next theorem is just an immediate consequence of this corollary:

Expressiveness Theorem 3. *$\mathcal{ALC}\mu$ involving solely terminologies comprising at most one concept introduction is at least as strong in expressive power as $\mathcal{ALC}\mu$. This holds for $\mathcal{ALC}\mu^-$ as well.*

We next compare both $\mathcal{ALC}\mu^-$ and full $\mathcal{ALC}\mu$ with the regular and ω-regular extension of $\mathcal{ALC}$ in their expressive power. For the **regular extension** of $\mathcal{ALC}$ see [Baader, 1991] or [Schild, 1991]. It additionally comprises the reflexive-transitive closure R^* of a role, the composition $R \circ S$ and union $R \sqcup S$ of two roles, the identity role *id*, as well as the role $R|C$ restricting the range of a role to a concept. The **ω-regular extension** of $\mathcal{ALC}$ extends its regular extension by the additional concept $\exists R^\omega$ which stipulates the existence of an infinite chain of the role R. It is worth mentioning that this language sometimes can be used to clarify the actual meaning of fixpoint terminologies. For instance, Streett [1985, page 364] mentioned the following equivalences:

$$\models \mu A.\{A \doteq C \sqcap \forall R{:}A\} \;\doteq\; (\forall R^*{:}C) \sqcap \neg\exists R^\omega$$

$$\models \nu A.\{A \doteq C \sqcup \exists R{:}A\} \;\doteq\; (\exists R^*{:}C) \sqcup \exists R^\omega$$

As usual, A has to be some concept name not appearing in C. According to Proposition 4, both equivalences can be carried over directly to the corresponding fixpoint terminologies:

$$\mu\{A \doteq C \sqcap \forall R{:}A\} \models A \doteq (\forall R^*{:}C) \sqcap \neg\exists R^\omega$$
$$\nu\{A \doteq C \sqcup \exists R{:}A\} \models A \doteq (\exists R^*{:}C) \sqcup \exists R^\omega$$

This shows that *some* concepts of the ω-regular extension of $\mathcal{ALC}$ are definable by syntactically monotone complex fixpoint terminologies of $\mathcal{ALC}$. The next theorem implies together with Expressiveness Theorem 1 that they are able to define *all* concepts of the *regular* extension of $\mathcal{ALC}$ and that they are even able to define concepts which are not equivalent to any concept of the regular extension of $\mathcal{ALC}$.

Expressiveness Theorem 4. *$\mathcal{ALC}\mu^-$ is strictly stronger in expressive power than the regular extension of $\mathcal{ALC}$, while $\mathcal{ALC}\mu$ is strictly stronger in expressive power than the ω-regular extension of $\mathcal{ALC}$.*

Proof. Consider the following equivalences which presuppose that A is some concept name not occurring in C:

$$\begin{aligned}
&\models \forall R{:}C &&\doteq \neg\exists R{:}\neg C\\
&\models \exists R^*{:}C &&\doteq \mu A.\{A \doteq C \sqcup \exists R{:}A\}\\
&\models \exists(R\circ S){:}C &&\doteq \exists R{:}\exists S{:}C\\
&\models \exists(R\sqcup S){:}C &&\doteq (\exists R{:}C)\sqcup(\exists S{:}C)^3\\
&\models \exists(R|C){:}D &&\doteq \exists R{:}(C\sqcap D)\\
&\models \exists id{:}C &&\doteq C\\
&\models \exists R^\omega &&\doteq \nu A.\{A \doteq \exists R{:}A\}
\end{aligned}$$

These equivalences can be used directly to prove by induction on the complexity of concepts of $\mathcal{ALC}\mu$ that $\mathcal{ALC}\mu$ is at least as strong in expressive power as the ω-regular extension of $\mathcal{ALC}$. As A does not occur in C, all but the last equivalence yield restricted concepts whenever C and D are restricted. The last equivalence, however, may yield concepts which are not restricted. For instance, $\exists(R\circ S^*)^\omega$ is equivalent to $\mu\boldsymbol{A}.\{\boldsymbol{A} \doteq \forall R{:}\nu B.\{B \doteq \boldsymbol{A} \sqcap \forall S{:}B\}\}$ which is not restricted. But this concerns solely the last of the above equivalences, so that $\mathcal{ALC}\mu^-$ is yet at least as strong in expressive power as the *regular* extension of $\mathcal{ALC}$. Now, according to Kozen [1983, Proposition 4.1], $\nu CN.\{CN \doteq \exists RN{:}CN\}$, which is equivalent to $\exists RN^\omega$, is not equivalent to any concept of the regular extension of $\mathcal{ALC}$. Finally, Niwinsky proved $\nu CN.\{CN \doteq \exists RN_1{:}CN \sqcap \exists RN_2{:}CN\}$ to be not equivalent to any concept of the ω-regular extension of $\mathcal{ALC}$ (see [Streett, 1985, Theorem 2.7]). □

[3] If we were concerned with linear length-boundedness, we could have taken the equivalent concept $\mu A.\{A \doteq (\exists R{:}B)\sqcup(\exists S{:}B), B \doteq C\}$ instead.

7 Computational Complexity

In what follows, we shall prove that as far as syntactically monotone terminologies of $\mathcal{ALC}$ are concerned, all three kinds of semantics essentially do not differ in the computational complexity of the corresponding subsumption relation. In fact, in each case subsumption is complete for deterministic exponential time and, moreover, there is a constant $c > 0$ such that in none of the cases it is computable in deterministic time 2^{cn}. It should be stressed that the lower exponential time bound actually improves the exponential time hardness result. This is due to that although it is known that problems which are hard for exponential time are not computable in polynomial time, it is not known whether for each of these problems there is a constant $c > 0$ such that the problem requires more than deterministic time 2^{cn}. To be more accurate, we shall investigate the computational complexity of the following three problems: For an arbitrary syntactically monotone terminology $\mathcal{T}$ of $\mathcal{ALC}$, and for an arbitrary axiom $C \doteq D$ of $\mathcal{ALC}$ decide whether (a) $\mathcal{T} \models C \doteq D$, (b) $\mu\mathcal{T} \models C \doteq D$, and (c) $\nu\mathcal{T} \models C \doteq D$. It will turn out that all three problems are complete for deterministic exponential time and, moreover, there is a constant $c > 0$ such that none of the problems is computable in deterministic time 2^{cn}. All these results hold even if $\mathcal{T}$ contains at most one concept introduction and $C \doteq D$ is a primitive concept introduction. In addition, we shall see that the entailment relation integrating all three kinds of semantics is also computable in deterministic exponential time. Here we mean the problem to decide whether $\mathcal{A} \cup \Gamma \models C \doteq D$, for arbitrary syntactically monotone complex fixpoint terminologies Γ of $\mathcal{ALC}$ and for arbitrary finite sets $\mathcal{A} \cup \{C \doteq D\}$ of axioms of $\mathcal{ALC}$.

Before entering into details, the reader should recall some basic notions of structural complexity theory. First of all, problems are usually represented in complexity theory as *sets*. The first of the problems mentioned above thus will be represented by the set of all tuples $\langle \mathcal{T}, C \doteq D \rangle$ such that $\mathcal{T}$ ranges over all syntactically monotone terminologies of $\mathcal{ALC}$ and $C \doteq D$ ranges over all axioms of $\mathcal{ALC}$ with $\mathcal{T} \models C \doteq D$. The following fundamental notion of structural complexity theory states intuitively that some set S is at least hard as another set T: T is called **polynomial time *m*-reducible** to S iff there is a function $\pi : T \to S$ computable in polynomial time such that for all x, $x \in T$ iff $\pi(x) \in S$. If there is additionally some constant $c > 0$ such that for all x, $|\pi(x)| \leq c|x|$, then T is said to be **polynomial time *lin*-reducible** to S. It can easily be seen that both polynomial time m-reducibility and polynomial time lin-reducibility are preorders, i.e., they constitute a reflexive and transitive relation on sets. Given a class $\mathcal{C}$, a set S is called **hard** for $\mathcal{C}$ iff every element of $\mathcal{C}$ is polynomial time m-reducible to S, and a set is **complete** for $\mathcal{C}$ iff it is hard for $\mathcal{C}$ and it is a member of $\mathcal{C}$. For any function t with $t(n) > n$, we denote with $\mathbf{DTIME}(t)$ the class of all sets accepted by deterministic Turing machines whose running time is bounded above by $t(n)$, for each input of length n. Similarly, for any function s with $s(n) \geq 1$, **DSPACE(s)** denotes the class of all sets accepted by deterministic Turing machines whose work space is bounded above by $s(n)$. We define **P** as $\bigcup_{i\geq 0}$ DTIME(n^i), **DEXT** as $\bigcup_{c\geq 0}$ DTIME(2^{cn}), **EXPTIME** as $\bigcup_{i\geq 0}$ DTIME(2^{n^i}), and **PSPACE** as $\bigcup_{i\geq 0}$ DSPACE(n^i). We shall make use the fact that EXPTIME is closed under polynomial time m-reducibility, whereas DEXT is closed under polynomial time lin-reducibility. That is, a set is a member of EXPTIME if it is polynomial time m-reducible to some set in EXPTIME and a set is a member of DEXT if it is polynomial time lin-reducible to some set in DEXT. Note, however, that DEXT is not closed under polynomial time m-reducibility. After all, for each function f, DTIME(f) is closed under complementation, i.e., a set is member of DTIME(f) iff its complement is a member of DTIME(f). This implies, for instance, that P = co-P and EXPTIME = co-EXPTIME and, therefore, any set is hard for EXPTIME iff it is hard for co-EXPTIME. According to the well-known linear speed-up theorem, it holds for each constant $c > 0$ that DTIME(2^{cn}) = $\bigcup_{d>0}$ DTIME($d2^{cn}$). For details the reader is referred to Chapter 3 of the excellent book of Balcázar *et al.* [1988].

For the complexity results to be presented, it is worth mentioning that P $\subseteq$ PSPACE $\subseteq$ EXPTIME and, moreover, P *strictly* included in DEXT which in turn is *strictly* included in EXPTIME. It is also known that the class of all sets acceptable in deterministic linear space, i.e., $\bigcup_{c>0}$ DSPACE(cn) is included in DEXT. It is not known, though, whether whole PSPACE is included in DEXT or vice versa. The only fact that *is* known is that PSPACE $\neq$ DEXT. For all these results the reader is referred to Theorem 2.8, Proposition 3.1, and Exercise 14 of Chapter 3.9 in [Balcázar *et al.*, 1988].

7.1 Lower Bounds

For the lower complexity bounds, we shall utilize a result due to Fischer and Ladner [1979]. It states roughly that accepting the set of coherent concepts of the regular extension of $\mathcal{ALC}$ is hard for EXPTIME and requires more than deterministic time $c^{n/\log n}$, for some constant $c > 1$, even if the concepts may contain solely at most one occurrence of * [Fischer and Ladner, 1979, Lemma 4.1 & Theorem 4.4]. Harel [1984] observed that Fischer and Ladner's proof also shows that this set is not acceptable in deterministic time 2^{cn}, for some constant $c > 0$ [Harel, 1984, Theorem 2.14]. Inspection of the Fischer and Ladner's proof immediately reveals that the syntactic form of the concepts can be restricted further:

Proposition 5. *The set of all coherent concepts of the form $C \sqcap \forall RN^*{:}D$ such that both C and D are concepts of $\mathcal{ALC}$ and RN is a role name, hence-*

forth denoted with **FL**, *is hard for* EXPTIME. *There is moreover a constant* $c > 0$ *such that* FL *is* not *a member of* DTIME(2^{cn}). *Both results hold even if C is solely composed of concept names, their negations, as well as $\sqcap$ and, additionally, at least one concept name occurs in C positively.*

First of all, we give a lower bound for computing the subsumption relation with respect to the descriptive semantics.

Complexity Theorem 1. *The set of all $\langle \mathcal{T}, CN \sqsubseteq C\rangle$ such that $\mathcal{T}$ ranges over all syntactically monotone terminologies of $\mathcal{ALC}$ and $CN \sqsubseteq C$ ranges over all concept introductions of $\mathcal{ALC}$ with $\mathcal{T} \models CN \sqsubseteq C$ is hard for* EXPTIME. *Moreover, there is a constant $c > 0$ such that this set is* not *a member of* DTIME(2^{cn}). *Both results hold even if $\mathcal{T}$ may contain at most one concept introduction.*

Proof. In what follows, we shall prove the following: Assume C and D are arbitrary concepts which do not contain any occurrence of the concept name A. It then holds that:

$$\models C \sqcap \forall RN^*{:}D \doteq \bot \quad \textit{iff} \quad A \sqsubseteq D \sqcap \forall RN{:}A \models A \sqsubseteq \neg C \tag{7}$$

This proves the *complement* of FL and the following set to be polynomial time *lin*-reducible to each other: The set of all $\langle\{A \sqsubseteq D \sqcap \forall RN{:}A\}, A \sqsubseteq \neg C\rangle$ such that C and D range over all concepts of $\mathcal{ALC}$ which do not contain any occurrence of A such that $A \sqsubseteq D \sqcap \forall RN{:}A \models A \sqsubseteq \neg C$. As FL is hard for EXPTIME, the the latter set is hard for EXPTIME as well. Now, if this set were a member of $\bigcap_{c>0}$ DTIME(2^{cn}), the complement of FL and thus FL itself would be elements of $\bigcap_{c>0}$ DTIME(2^{cn}). This would contradict, however, Proposition 5 which states that for some constant $c > 0$, FL is *not* a member of DTIME(2^{cn}).

We shall prove both directions of (7) by contraposition. For the if-part, suppose $C \sqcap \forall RN^*{:}D$ is not equivalent to $\bot$. That is, there is at least one interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\rangle$ such that $C^{\mathcal{I}} \cap (\forall RN^*{:}D)^{\mathcal{I}} \neq \bot^{\mathcal{I}} = \emptyset$. Clearly, $C^{\mathcal{I}} \cap (\forall RN^*{:}D)^{\mathcal{I}} \neq \emptyset$ holds exactly when

$$(\forall RN^*{:}D)^{\mathcal{I}} \not\subseteq \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} = \neg C^{\mathcal{I}} \tag{8}$$

Suppose N is the set of all concept and role names occurring in $C \sqcap \forall RN^*{:}D$. Consider some interpretation $\mathcal{J} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{J}}\rangle$ which is N-compatible with $\mathcal{I}$ such that $A^{\mathcal{J}} = (\forall RN^*{:}D)^{\mathcal{I}}$. As A occurs neither in C nor in D, there is actually such an interpretation $\mathcal{J}$. It will turn out that $\mathcal{J}$ is a model of $A \sqsubseteq D \sqcap \forall RN{:}A$, but it is not model of $A \sqsubseteq \neg C$, so that $\{A \sqsubseteq D \sqcap \forall RN{:}A\}$ does not entail $A \sqsubseteq \neg C$. The fact that $\mathcal{J}$ is no model of $A \sqsubseteq \neg C$ is an immediate consequence of the assumption that $A^{\mathcal{J}} = (\forall RN^*{:}D)^{\mathcal{I}}$ together with (8) which states that $(\forall RN^*{:}D)^{\mathcal{I}} \not\subseteq \neg C^{\mathcal{I}}$. Since $\mathcal{J}$ is N-compatible with $\mathcal{I}$, the functions $.^{\mathcal{I}}$ and $.^{\mathcal{J}}$ map D to the same subset of $\Delta^{\mathcal{I}}$ and RN to the same binary relation over $\Delta^{\mathcal{I}}$. Together with the assumption that $A^{\mathcal{J}} = (\forall RN^*{:}D)^{\mathcal{I}}$ this proves that $\mathcal{J}$ is in fact a model of $A \sqsubseteq D \sqcap \forall RN{:}A$:

$$\begin{array}{rl} & A^{\mathcal{J}} \\ = & (\forall RN^*{:}D)^{\mathcal{I}} \\ = & (D \sqcap \forall RN{:}\forall RN^*{:}D)^{\mathcal{I}} \\ = & D^{\mathcal{I}} \cap \{d \in \Delta^{\mathcal{I}} : RN^{\mathcal{I}}(d) \subseteq (\forall RN^*{:}D)^{\mathcal{I}}\} \\ = & D^{\mathcal{J}} \cap \{d \in \Delta^{\mathcal{I}} : RN^{\mathcal{J}}(d) \subseteq A^{\mathcal{J}}\} \\ = & (D \sqcap \forall RN{:}A)^{\mathcal{J}} \end{array}$$

To prove the only-if-part of (7), assume $\{A \sqsubseteq D \sqcap \forall RN{:}A\}$ does not entail $A \sqsubseteq \neg C$. That is, there is at least one interpretation, say $\mathcal{I} = \langle \Delta^{\mathcal{I}}, .^{\mathcal{I}} \rangle$, which is a model of $A \sqsubseteq D \sqcap \forall RN{:}A$, but which is no model of $A \sqsubseteq \neg C$. The latter means that $A^{\mathcal{I}} \not\subseteq \neg C^{\mathcal{I}}$, i.e., $A^{\mathcal{I}} \cap C^{\mathcal{I}} \neq \emptyset$, whereas the former means $A^{\mathcal{I}}$ must be a subset of $(D \sqcap \forall RN{:}A)^{\mathcal{I}}$. It is folklore that $(\forall RN{:}C_1)^{\mathcal{I}}$ is a subset of $(\forall RN{:}C_2)^{\mathcal{I}}$ if $C_1^{\mathcal{I}}$ is a subset of $C_2^{\mathcal{I}}$. Using this observation it is easy to see that $A^{\mathcal{I}}$ is a subset of $(D \sqcap \forall RN{:}D)^{\mathcal{I}}$:

$$\begin{array}{rl} & A^{\mathcal{I}} \\ \subseteq & (D \sqcap \forall RN{:}A)^{\mathcal{I}} \\ = & D^{\mathcal{I}} \cap (\forall RN{:}A)^{\mathcal{I}} \\ \subseteq & D^{\mathcal{I}} \cap (\forall RN{:}(D \sqcap \forall RN{:}A))^{\mathcal{I}} \\ = & (D \sqcap \forall RN{:}D)^{\mathcal{I}} \cap (\forall RN{:}\forall RN{:}A)^{\mathcal{I}} \\ \subseteq & (D \sqcap \forall RN{:}D)^{\mathcal{I}} \end{array}$$

Induction on n proves that for any natural number n, $A^{\mathcal{I}}$ is a subset of $(D \sqcap (\forall RN^1{:}D) \ldots \sqcap (\forall RN^n{:}D))^{\mathcal{I}}$. As one might suspect, $\forall RN^n{:}D$ abbreviates $\forall RN^{n-1}{:}D$ if $n > 1$, whereas $\forall RN^1{:}D$ is $\forall RN{:}D$. This means, $A^{\mathcal{I}}$ is a subset of $(\forall RN^*{:}D)^{\mathcal{I}}$ and, therefore, $A^{\mathcal{I}} \cap C^{\mathcal{I}}$ is subset of $(\forall RN^*{:}D)^{\mathcal{I}} \cap C^{\mathcal{I}}$. As $A^{\mathcal{I}} \cap C^{\mathcal{I}}$ is assumed to be nonempty, $(\forall RN^*{:}D)^{\mathcal{I}} \cap C^{\mathcal{I}}$ must be nonempty too. But this is just to say that $C \sqcap \forall RN^*{:}D$ is coherent. □

Next, we give lower complexity bounds for the the least and the greatest fixpoint semantics.

Complexity Theorem 2. *The set of all $\langle \mu\mathcal{T}, CN \sqsubseteq C \rangle$ such that $\mathcal{T}$ ranges over all syntactically monotone terminologies of $\mathcal{ALC}$ and $CN \sqsubseteq C$ ranges over all concept introductions of $\mathcal{ALC}$ with $\mu\mathcal{T} \models CN \sqsubseteq C$ is hard for* EXPTIME. *There is moreover a constant $c > 0$ such that this set is* not *a member of* DTIME(2^{cn}). *Both results hold even if $\mathcal{T}$ may contain at most one concept introduction. The corresponding statements hold for for greatest fixpoint terminologies as well.*

Proof. We have already seen that $\mu\{A \doteq C \sqcup \exists RN{:}A\}$ has exactly the same models as $A \doteq \exists RN^*{:}C$ and that $\nu\{A \doteq C \sqcap \forall RN{:}A\}$ has exactly the same models as $A \doteq \forall RN^*{:}C$. The only condition is that A does not occur in C. This means, if A occurs neither in C nor in D, then it holds that:

$$\begin{array}{lll} \models C \sqcap \forall RN^*{:}D \doteq \bot & \textit{iff} & \nu\{A \doteq D \sqcap \forall RN{:}A\} \models C \sqcap A \doteq \bot \\ & \textit{iff} & \mu\{A \doteq \neg D \sqcup \exists RN{:}A\} \models C \sqcap \neg A \doteq \bot \end{array}$$

To proceed, observe that $C \sqcap A \doteq \bot$ and the *primitive* concept introduction $A \sqsubseteq \neg C$ have exactly the same models. Similarly, if C is of the form $CN \sqcap C'$, $C \sqcap \neg A \doteq \bot$ and the *primitive* concept introduction $CN \sqsubseteq A \sqcup \neg C'$ have exactly the same models too. To summarize, if A occurs neither in C nor in D, and if C is of the form $CN \sqcap C'$, then it holds that:

$$\begin{array}{lll} \models C \sqcap \forall RN^*{:}D \doteq \bot & \textit{iff} & \nu\{A \doteq D \sqcap \forall RN{:}A\} \models A \sqsubseteq \neg C \\ & \textit{iff} & \mu\{A \doteq \neg D \sqcup \exists RN{:}A\} \models CN \sqsubseteq A \sqcup \neg C' \end{array}$$

This means, the sets whose lower complexity bounds we are about to prove are polynomial time *lin*-reducible to the *complement* of FL and vice versa, at least when taking the restriction into consideration which is mentioned in Proposition 5. As FL is hard for EXPTIME, these sets are hard for EXPTIME as well. Furthermore, if they were a member of $\bigcap_{c>0} \mathrm{DTIME}(2^{cn})$, the complement of FL and thus FL itself would be elements of $\bigcap_{c>0} \mathrm{DTIME}(2^{cn})$. This would contradict, however, Proposition 5 which states that for some constant $c > 0$, FL is *not* a member of $\mathrm{DTIME}(2^{cn})$. □

7.2 Upper Bounds

Streett and Emerson [1984, 1989] gave an elementary upper time bound for accepting the set of coherent concepts of full $\mathcal{ALC}\mu$. Vardi and Wolper [1984] show that the set of coherent concepts of $\mathcal{ALC}\mu^-$ is a member of EXPTIME. The next theorem generalize Vardi and Wolper's result to subsumption with respect to finite sets of axioms of $\mathcal{ALC}\mu^-$.

Complexity Theorem 3. *The set of all $\langle \mathcal{A}, C \doteq D\rangle$ such that $\mathcal{A} \cup \{C \doteq D\}$ ranges over all finite sets of axioms of $\mathcal{ALC}\mu^-$ with $\mathcal{A} \models C \doteq D$ is a member of* EXPTIME.

Proof. As already mentioned, Vardi and Wolper [1984, Theorem 3] proved the claim to hold for $\mathcal{A}$ being the empty set. In the same paper they also showed that each concept C of $\mathcal{ALC}\mu^-$ is coherent iff there is a tree interpretation $\mathcal{I} = \langle \Delta^\mathcal{I}, .^\mathcal{I}\rangle$ such that the empty word λ is a element of $C^\mathcal{I}$ [Theorem 2]. According to Vardi and Wolper [1986, Page 197], a **tree interpretation** is an interpretation $\langle \Delta^\mathcal{I}, .^\mathcal{I}\rangle$ such that:

1. For some finite alphabet Σ, $\Delta^{\mathcal{I}}$ is a nonempty subset of all words over Σ.

2. For every $w \in \Sigma^*$ and for every $a \in \Sigma$, $wa \in \Delta^{\mathcal{I}}$ only if $w \in \Delta^{\mathcal{I}}$.

3. For each role name RN and all $w, w' \in \Sigma^*$, $\langle w, w' \rangle \in RN^{\mathcal{I}}$ only if for some $a \in \Sigma$, $w' = wa$ and, additionally, for each other role name RN', $\langle w, w' \rangle \notin RN'^{\mathcal{I}}$.

This ensures that any terminological axiom $C \doteq D$ can be internalized within $\mathcal{ALC}\mu^-$ using the technique introduced independently by Baader [1991] and by Schild [1991]. This technique utilizes the concept $\forall_{\mathcal{R}}{:}C$ defined as follows if $\mathcal{R} = \{RN_i : 1 \leq i \leq n\}$ is a finite set of role names:

$$\forall_{\mathcal{R}}{:}C \stackrel{\text{def}}{=} \nu A.\{A \doteq C \sqcap \forall RN_1{:}A ... \sqcap \forall RN_n{:}A\}$$

The concept name A must not occur in C. We have already seen that the concepts $\nu A.\{A \doteq C \sqcap \forall RN{:}A\}$ and $\forall RN^*{:}C$ are in fact equivalent. In analogy to this equivalence, $\forall_{\mathcal{R}}{:}C$ is equivalent to $\forall(RN_1 ... \sqcup RN_n)^*{:}C$, if $\mathcal{R}$ is as above. Suppose $\overline{C \doteq D}$ abbreviates the concept $(\neg C \sqcup D) \sqcap (\neg D \sqcup C)$. Then for each *tree* interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, it holds that $\mathcal{I}$ is a model of $C \doteq D$ iff $\lambda \in (\forall_{\mathcal{R}}{:}\overline{C \doteq D})^{\mathcal{I}}$, provided $\mathcal{R}$ is the set of all role names appearing in C and D. In addition, it holds that the empty word λ is a element of $(\forall_{\mathcal{R}}{:}\overline{C \doteq D})^{\mathcal{I}}$ iff $(\forall_{\mathcal{R}}{:}\overline{C \doteq D})^{\mathcal{I}}$ is the full domain $\Delta^{\mathcal{I}}$, so that $C \doteq D \models C' \doteq D'$ iff $\models (\forall_{\mathcal{R}}{:}\overline{C \doteq D}) \sqsubseteq \overline{C' \doteq D'}$. This internalization is clearly computable in polynomial time and it does not increase the size of the involved axioms more than linearly. Moreover, it preserves the restrictedness of the involved concepts. Induction on the cardinality of $\mathcal{A}$ shows that the set of all tuples $\langle \mathcal{A}, C \doteq D \rangle$ such that $\mathcal{A} \cup \{C \doteq D\}$ ranges over all finite sets $\mathcal{A}$ of axioms of $\mathcal{ALC}\mu^-$ with $\mathcal{A} \models C \doteq D$ is polynomial time *lin*-reducible to the same set but with $\mathcal{A}$ restricted to be empty. As the latter is a member of EXPTIME, the former must be an element of EXPTIME too. □

It should be stressed that according to Representation Theorem 1, the last theorem actually gives an upper complexity bound for computing the subsumption relation which integrates all three kinds of semantics.

Corollary 3. *The set of all $\langle \mathcal{A} \cup \Gamma, C \doteq D \rangle$ such that Γ ranges over all syntactically monotone complex fixpoint terminologies of $\mathcal{ALC}$ and $\mathcal{A} \cup \{C \doteq D\}$ ranges over all finite sets of axioms of $\mathcal{ALC}\mu^-$ with $\mathcal{A} \cup \Gamma \models C \doteq D$ is a member of* EXPTIME.

8 Conclusion

We have investigated terminological cycles in the terminological standard logic $\mathcal{ALC}$ with the only restriction that recursively defined concepts must occur in their definition positively. This restriction, called syntactic monotonicity, ensures the existence of least and greatest fixpoint models. It turned out that as far as syntactically monotone terminologies of $\mathcal{ALC}$ are concerned, the descriptive semantics as well as the least and greatest fixpoint semantics do not differ in the computational complexity of the corresponding subsumption relation. In fact, in each case subsumption is complete for deterministic exponential time and, moreover, there is a constant $c > 0$ such that in none of the cases it is computable in in deterministic time 2^{cn}. These complexity results significantly improve those of Baader [1990] in the sense that he investigated only syntactically monotone terminologies in a very small sublanguage of ours. The concept language he considered comprises neither concept disjunction, concept negation, nor existential value restrictions of the form $\exists R{:}C$. We also showed that the expressive power of finite sets of syntactically monotone terminologies of $\mathcal{ALC}$ is the very same for least and greatest fixpoint semantics. We have moreover seen that in both cases they are *strictly* stronger in expressive power than $\mathcal{ALC}$ augmented by regular role expressions. This contrasts a result of Baader [1990] who proved that syntactically monotone least and greatest fixpoint terminologies of the restricted language he considered can define solely concepts of the regular extension of $\mathcal{ALC}$ [Baader, 1990]. Our results clarify the ongoing discussion on the adequate semantics for terminological cycles. They show that for none of the three kinds of semantics is preferable in terms of computational complexity of the corresponding subsumption relation. Moreover, our results show that neither the least nor the greatest fixpoint semantics is preferable in terms of expressive power. We obtained these results by a direct correspondence to the so-called propositional μ-calculus which allows to express least and greatest fixpoints explicitly. $\mathcal{ALC}$ augmented by these fixpoint operators, called $\mathcal{ALC}\mu^-$, provides a unifying framework for all three kinds of semantics.

The correspondence, however, yields further results. For instance, there are already tableau-based algorithms and even workbenches for $\mathcal{ALC}\mu^-$, at least as far as only terminologies comprising at most one concept introduction are involved [Huhn and Niebert, 1993].[4] Concerning the restriction to terminologies comprising at most one concept introduction, recall that we have seen that this restriction does not decrease the expressive power of $\mathcal{ALC}\mu^-$.

[4] Huhn and Niebert [1993] actually gave only a tableau-based algorithm for a slightly restricted version of $\mathcal{ALC}\mu^-$.

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