Capturing Graded Knowledge and Uncertainty in a Modalized Fragment of OWL

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Abstract: Natural language statements uttered in diagnosis (e.g., in medicine), but more general in daily life are usually *graded*, i.e., are associated with a degree of uncertainty about the validity of an assessment and is often expressed through specific *verbs*, *adverbs*, or *adjectives* in natural language. In this paper, we look into a *representation* of such graded statements by presenting a simple non-standard modal logic which comes with a set of modal operators, directly associated with the words indicating the uncertainty and interpreted through confidence intervals in the model theory. We complement the model theory by a set of RDFS-/OWL 2 RL-like entailment (*if-then*) rules, acting on the syntactic representation of modalized statements. Our interest in such a formalization is related to the use of OWL as the *de facto* language in today's ontologies and its weakness to represent and reason about assertional knowledge that is *uncertain* or that changes over time.

1 INTRODUCTION

Medical natural language statements uttered by physicians or other health professionals and found in medical examination letters are usually *graded*, i.e., are associated with a degree of uncertainty about the validity of a medical assessment. This uncertainty is often expressed through specific *verbs, adverbs, adjectives*, or even *phrases* in natural language which we will call <u>gradation words</u> ($\approx$ linguistic hedges); e.g., *Dr. X <u>suspects</u> that Y suffers from Hepatitis* or *The patient <u>probably</u> has Hepatitis* or *(The diagnosis of) Hepatitis is <u>confirmed</u>*.

In this paper, we look into a representation of such graded statements by presenting a simple non-standard modal logic which comes with a small set of *partially-ordered modal operators*, directly associated with the words indicating the uncertainty and interpreted through *confidence intervals* in the model theory. The work presented here addresses modalized propositional formulae in negation normal form which can be seen as a canonical representation of natural language sentences of the above form (a kind of a *controlled natural language*).

Our interest in such a formalization is related to the use of OWL in our projects as the *de facto standard* for (medical) ontologies today (to represent structural/terminological knowledge) and its *weakness* to represent and reason about assertional knowledge that is uncertain (Schulz et al., 2014) or that changes over time (Krieger, 2012). There are two principled ways to address such a restriction: *either* by sticking with the existing formalism (viz., OWL) and trying to find an encoding that still enables some useful forms of reasoning (Schulz et al., 2014); *or* by deviating from a defined standard in order to arrive, at best, at an easier, intuitive, and less error-prone representation (Krieger, 2012).

Here, we follow the latter avenue, but employ and extend the standard entailment rules from (Hayes, 2004; ter Horst, 2005; Motik et al., 2012) for positive binary relation instances in RDFS and OWL towards modalized *n*-ary relation instances, including negation. These entailment rules talk about, e.g., subsumption, class membership, or transitivity, and have been found useful in many applications. The proposed solution has been implemented for the binary relation case (extended triples, quads) in *HFC* (Krieger, 2013), a forward chaining engine that builds Herbrand models which are compatible with the open-world view underlying OWL.

Our approach is clearly not restricted to medical statements, but is applicable to graded statements in general, e.g., in technical diagnosis (*the engine is <u>probably</u> overheated*) or in everyday conversation (*I'm <u>pretty sure</u> that Joe has signed a contract with*

Foo Inc.), involving *trust* (*I'm not an expert, but ...*) which can be seen as the common case (contrary to true *universal* statements).

2 OWL VS. MODALIZED REPRESENTATION

We note here that the names of our *initial* modal operators were inspired by the *qualitative information parts* of diagnostic statements from (Schulz et al., 2014) as shown in Figure 1.

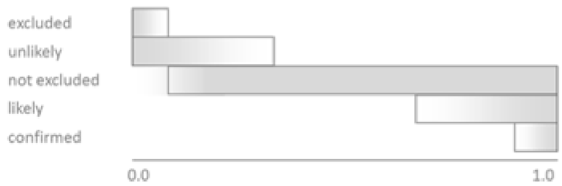

Figure 1: Schematic mappings of the qualitative information parts *excluded* (E), *unlikely* (U), *not excluded* (N), *likely* (L), and *confirmed* (C) to *confidence intervals*. Picture taken from (Schulz et al., 2014).

These qualitative parts were used in medical statements about, e.g., liver inflammation with varying levels of detail (Schulz et al., 2014) in order to infer, e.g., **if** *Hepatitis is confirmed* **then** *Hepatitis is likely* but **not** *Hepatitis is unlikely*. And **if** *Viral Hepatitis B is confirmed*, **then** both *Viral Hepatitis is confirmed* **and** *Hepatitis is confirmed* (generalization). Things "turn around" when we look at the adjectival modifiers *excluded* and *unlikely*: **if** *Hepatitis is excluded* **then** *Hepatitis is unlikely*, but **not** *Hepatitis is not excluded*. Furthermore, **if** *Hepatitis is excluded*, **then** both *Viral Hepatitis is excluded* **and** *Viral Hepatitis B is excluded* (specialization).

(Schulz et al., 2014) consider five OWL encodings, from which only two were able to fully reproduce the *plausible* inferences for the above Hepatitis use case. The encodings in (Schulz et al., 2014) were quite *cumbersome* as the primary interest was to stay within the limits of the underlying calculus. Besides coming up with complex encodings, only minor forms of reasoning were possible, viz., subsumption reasoning. Furthermore, each combination of disease and qualitative information part required a *new* OWL class definition/new class name, and there exist a lot of them!

These disadvantages are a result of two conscious decisions: OWL only provides unary and binary relations (concepts and roles) and comes up with a (mostly) fixed set of entailment/tableaux rules.

In our approach, however, the *qualitative information parts* from Figure 1 are first class citizens of the object language (the modal operators) and *diagnostic statements* from the Hepatitis use case are expressed through the binary property $\mathsf{suffersForm}$ between p (patients, people) and d (diseases, diagnoses). The plausible inferences are then simply a *byproduct* of the *instantiation* of the entailment rule schemas (G) from Section 5.1, and (S1) and (S0) from Section 5.2 for property $\mathsf{suffersForm}$ (the rule variables are universally quantified; $\top$ = *universal truth*; C = *confirmed*; L = *likely*), e.g.,

(S1) $\mathsf{ViralHepatitisB} \sqsubseteq \mathsf{ViralHepatitis} \wedge \top\mathsf{ViralHepatitisB}(d) \rightarrow \top\mathsf{ViralHepatitis}(d)$

(G) $C\mathsf{suffersFrom}(p,d) \rightarrow L\mathsf{suffersFrom}(p,d)$

Two things are worth mentioning here. *Firstly*, not only OWL properties can be graded, such as $C\mathsf{suffersFrom}(p,d)$ (= *it is confirmed that p suffers from d*), but also class membership, e.g., $C\mathsf{ViralHepatitisB}(d)$ (= *it is confirmed that d is of type Viral Hepatitis B*). As the original OWL example from (Schulz et al., 2014) can not make use of any modals, we employ the special modal $\top$ here: $\top\mathsf{ViralHepatitisB}(d)$. *Secondly*, modal operators are only applied to assertional knowledge (the ABox in OWL)—neither TBox nor RBox axioms are being affected by modals in our approach, as they are supposed to express universal truth.

3 CONFIDENCE AND CONFIDENCE INTERVALS

We address the *confidence* of an asserted (medical) statement (Schulz et al., 2014) through *graded* modalities applied to propositional formulae: E (*excluded*), U (*unlikely*), N (*not excluded*), L (*likely*), and C (*confirmed*). For various (technical) reasons, we add a *wildcard* modality ? (*unknown*), a complementary *failure* modality ! (*error*), plus two further modalities to syntactically state definite truth and falsity: $\top$ (*true*) and $\bot$ (*false*).[1]

Let $\triangle$ now denotes the set of all modalities:

$$\triangle := \{?, !, \top, \bot, E, U, N, L, C\}$$

A *measure function*

$$\mu : \triangle \mapsto [0,1] \times [0,1]$$

[1] We also call $\top$ and $\bot$ *propositional* modals as they lift propositional statements to the modal domain. We refer to ? and ! as *completion* modals since they complete the modal hierarchy by adding unique most general and most specific elements (see Section 4.3).

is a mapping which returns the associated *confidence interval* $\mu(\delta) = [l,h]$ for a modality from $\delta \in \triangle$ ($l \leq h$). We presuppose that

- $\mu(?) = [0,1]$
- $\mu(\top) = [1,1]$
- $\mu(\perp) = [0,0]$
- $\mu(!) = \emptyset$[2]

In addition, we define two disjoint subsets of $\triangle$, called

$$\underline{1} := \{\top, C, L, N\}$$

and

$$\underline{0} := \{\perp, E, U\}$$

and again make a presupposition: the confidence intervals for modals from $\underline{1}$ *end* in 1, whereas the confidence intervals for $\underline{0}$ modals always *start* with 0. It is worth noting that we do *not* make use of μ in the syntax of the modal language (for which we employ the modalities from $\triangle$), but in the semantics when dealing with the satisfaction relation of the model theory (see Section 4).

We have talked about *confidence intervals* now several times without saying what we actually mean by this. Suppose that a physician says that it is *confirmed* (= C) that patient p suffers from disease d, for a set of observed symptoms (or evidence) $S = \{S_1,\ldots,S_k\}$: C*suffersFrom*(p,d).

Assuming that a different patient p' shows the same symptoms S (and only S, and perhaps further symptoms which are, however, *independent* from S), we would assume that the same doctor would diagnose C*suffersFrom*(p',d).

Even an other, but similar trained physician is supposed to grade the two patients *similarly*. This similarity which originates from patients showing the same symptoms and from physicians being taught at the same medical school is addressed by confidence *intervals* and not through a *single* (posterior) probability, as there are still variations in diagnostic capacity and daily mental state of the physician. By using intervals (instead of single values), we can usually reach a consensus among people upon the *meaning* of gradation words, even though the low/high values of the confidence interval for, e.g., *confirmed* might depend on the context.

Being a bit more theoretic, we define a *confidence interval* as follows. Assume a *Bernoulli experiment* (Krengel, 2003) that involves a large set of n patients P, sharing the same symptoms S. W.r.t. our example, we would like to know whether *suffersFrom*(p,d) or $\neg$*suffersFrom*(p,d) is the case for every patient $p \in P$, sharing S. Given a Bernoulli trials sequence $\vec{X} = (X_1,\ldots,X_n)$ with indicator random variables $X_i \in \{0,1\}$ for a patient sequence $(p_1,\ldots,p_n)$, we can approximate the *expected value* E for *suffersFrom* being *true*, given disease d and background symptoms S by the *arithmetic mean* A:

$$\mathrm{E}[\vec{X}] \approx \mathrm{A}[\vec{X}] = \frac{\sum_{i=1}^{n} X_i}{n}$$

Due to the *law of large numbers*, we expect that if the number of elements in a trials sequence goes to infinity, the arithmetic mean will coincide with the expected value:

$$\mathrm{E}[\vec{X}] = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n}$$

Clearly, the arithmetic mean for each new *finite* trials sequence is different, but we can try to *locate* the expected value within an interval around the arithmetic mean:

$$\mathrm{E}[\vec{X}] \in [\mathrm{A}[\vec{X}] - \varepsilon_1, \mathrm{A}[\vec{X}] + \varepsilon_2]$$

For the moment, we assume $\varepsilon_1 = \varepsilon_2$, so that $\mathrm{A}[\vec{X}]$ is in the center of this interval which we will call from now on *confidence interval*.

Coming back to our example and assuming $\mu(C) = [0.9,1]$, C*suffersFrom*(p,d) can be read as being true in 95% of all cases *known* to the physician, involving patients p potentially having disease d and sharing the same prior symptoms (evidence) $S_1,\ldots,S_k$:

$$\frac{\sum_{p\in P} \mathrm{Prob}(\textit{suffersFrom}(p,d)|S)}{n} \approx 0.95$$

The variance of $\pm 5\%$ is related to varying diagnostic capabilities between (comparative) physicians, daily mental form, undiscovered important symptoms or examinations which have not been carried out (e.g., lab values), or perhaps even by the physical stature of the patient (crooked vs. upright) which unconsciously affects the final diagnosis, etc, as elaborated above. Thus the individual modals from $\triangle$ express (via μ) different forms of the physician's *confidence*, depending on the set of already acquired symptoms as (potential) explanations for a specific disease.

4 MODEL THEORY AND NEGATION NORMAL FORM

Let $\mathcal{C}$ denote the set of constants that serve as the arguments of a relation instance. For instance, in

[2]Recall that intervals are (usually infinite) sets of real numbers, together with an ordering relations (e.g., $<$ or $\leq$) over the elements, thus $\emptyset$ is a perfect, although degraded interval.

an RDF/OWL setting, $\mathcal{C}$ would exclusively consist of XSD atoms, blank nodes, and URIs/IRIs. In order to define basic n-ary propositional formulae (ground atoms), let $p(\vec{c})$ abbreviates $p(c_1,\ldots,c_n)$, for $c_1,\ldots,c_n \in C$, given $length(\vec{c}) = n$. In case the number of arguments does not matter, we sometimes simply write p, instead of, e.g., $p(c,d)$ or $p(\vec{c})$. As before, we assume $\triangle = \{?,!,\top,\bot,E,U,N,L,C\}$. We inductively define the set of *well-formed formulae* ϕ of our modal language as follows:

$$\phi ::= p(\vec{c}) \mid \neg\phi \mid \phi \wedge \phi' \mid \phi \vee \phi' \mid \triangle\phi$$

4.1 Simplification and Normal Form

We now syntactically *simplify* the set of well-formed formulae ϕ by restricting the uses of *negation* and *modalities* to the level of propositional letters π:

$$\pi ::= p(\vec{c}) \mid \neg p(\vec{c})$$
$$\phi ::= \pi \mid \triangle\pi \mid \phi \wedge \phi' \mid \phi \vee \phi'$$

The design of this language is driven by two main reasons: *firstly*, we want to effectively implement the logic (in our case, in *HFC*), and *secondly*, the application of the below semantic-preserving simplification rules in an offline pre-processing step makes the implementation easier and guarantees a more efficient runtime system. To address negation, we first need the notion of a *complement* modal δ^{C} for every $\delta \in \triangle$, where

$$\mu(\delta^{\mathsf{C}}) := \mu(\delta)^{\mathsf{C}} = \mu(?) \setminus \mu(\delta) = [0,1] \setminus \mu(\delta)$$

I.e., $\mu(\delta^{\mathsf{C}})$ is defined as the complementary interval of $\mu(\delta)$ (within the bounds of $[0,1]$, of course). For example, E and N (*excluded, not excluded*) or ? and ! (*unknown, error*) are already existing complementary modals.

We also require *mirror* modals δ^{M} for every $\delta \in \triangle$ whose confidence interval $\mu(\delta^{\mathsf{M}})$ is derived by "mirroring" $\mu(\delta)$ to the opposite side of the confidence interval, either to the left or to the right:

$$\textbf{if}\ \mu(\delta) = [l,1]\ \textbf{then}\ \mu(\delta^{\mathsf{M}}) := [0,1-l]$$
$$\textbf{if}\ \mu(\delta) = [0,h]\ \textbf{then}\ \mu(\delta^{\mathsf{M}}) := [1-h,1]$$

It is easy to see that these two equations can be unified and *generalized*[3]:

$$\textbf{if}\ \mu(\delta) = [l,h]\ \textbf{then}\ \mu(\delta^{\mathsf{M}}) := [1-h,1-l]$$

For example, E and C (*excluded, confirmed*) or $\top$ and $\bot$ (*top, bottom*) are mirror modals. In order to transform ϕ into its *negation normal form*, we need to apply simplification rules a finite number of times (until rules are no longer applicable). We depict those rules by using the $\vdash$ relation, read as *formula* $\vdash$ *simplified formula* (ε = empty word):

1. $?\phi \vdash \varepsilon$ % *$?\phi$ is not informative at all*
2. $\neg\neg\phi \vdash \phi$
3. $\neg(\phi \wedge \phi') \vdash \neg\phi \vee \neg\phi'$
4. $\neg(\phi \vee \phi') \vdash \neg\phi \wedge \neg\phi'$
5. $\neg\triangle\phi \vdash \triangle^{\mathsf{C}}\phi$ (example: $\neg E\phi = E^{\mathsf{C}}\phi = N\phi$)
6. $\triangle\neg\phi \vdash \triangle^{\mathsf{M}}\phi$ (example: $E\neg\phi = E^{\mathsf{M}}\phi = C\phi$)

Clearly, the mirror modals δ^{M} ($\delta \in \triangle$) are not necessary as long as we explicitly allow for negated statements (which we do), and thus case 6 can, in principle, be dropped.

What is the result of simplifying $\triangle(\phi \wedge \phi')$ and $\triangle(\phi \vee \phi')$? Let us start with the former case and consider as an example the statement about an engine that *a mechanical failure m* <u>*and*</u> *an electrical failure e is* <u>*confirmed*</u>: $C(m \wedge e)$. It seems *plausible* to simplify this expression to $Cm \wedge Ce$. Commonsense tells us furthermore that neither Em nor Ee is compatible with this description (we should be alarmed if, e.g., both Cm and Em happen to be the case).

Now consider the "opposite" statement $E(m \wedge e)$ which must *not* be rewritten to $Em \wedge Ee$, as *either* Cm *or* Ce is well *compatible* with $E(m \wedge e)$. Instead, we rewrite this kind of "negated" statement as $Em \vee Ee$, and this works fine with either Cm or Ce.

In order to address the other modal operators, we generalize these *plausible* inferences by making a distinction between $\underline{0}$ and $\underline{1}$ modals (cf. Section 3):

7a. $\underline{0}(\phi \wedge \phi') \vdash \underline{0}\phi \vee \underline{0}\phi'$

7b. $\underline{1}(\phi \wedge \phi') \vdash \underline{1}\phi \wedge \underline{1}\phi'$

Let us now focus on disjunction inside the scope of a modal operator. As we do allow for the full set of Boolean operators, we are allowed to deduce

8. $\triangle(\phi \vee \phi') \vdash \triangle(\neg(\neg(\phi \vee \phi'))) \vdash \triangle(\neg(\neg\phi \wedge \neg\phi')) \vdash \triangle^{\mathsf{M}}(\neg\phi \wedge \neg\phi')$

This is, again, a conjunction, so we apply schemas 7a and 7b, giving us

8a. $\underline{0}(\phi \vee \phi') \vdash \underline{0}^{\mathsf{M}}(\neg\phi \wedge \neg\phi') \vdash \underline{1}(\neg\phi \wedge \neg\phi') \vdash \underline{1}\neg\phi \wedge \underline{1}\neg\phi' \vdash \underline{1}^{\mathsf{M}}\phi \wedge \underline{1}^{\mathsf{M}}\phi' \vdash \underline{0}\phi \wedge \underline{0}\phi'$

8b. $\underline{1}(\phi \vee \phi') \vdash \underline{1}^{\mathsf{M}}(\neg\phi \wedge \neg\phi') \vdash \underline{0}(\neg\phi \wedge \neg\phi') \vdash \underline{0}\neg\phi \vee \underline{0}\neg\phi' \vdash \underline{0}^{\mathsf{M}}\phi \vee \underline{0}^{\mathsf{M}}\phi' \vdash \underline{1}\phi \vee \underline{1}\phi'$

Note how the modals from $\underline{0}$ in 7a and 8a act as a kind of *negation* operator to turn the logical operators into their counterparts, similar to de *Morgan's law*.

The final case considers two consecutive modals:

[3] This construction procedure comes in handy when dealing with *in-the-middle* modals, such as *fifty-fifty* or *perhaps*, whose confidence intervals neither touch 0 nor 1. Such modals have a *real* background in (medical) diagnosis.

9. $\delta' \delta'' \phi \vdash (\delta' \circ \delta'')\phi$

We interpret the $\circ$ operator as a kind of *function composition*, leading to a new modal δ which is the result of $\delta' \circ \delta''$. We take a liberal stance here of what the result is, but indicate that it depends on the domain and, again, plausible inferences we like to capture. The $\circ$ operator will probably be different from the related operation $\odot$ which is used in Section 5.3.4.

4.2 Model Theory

In the following, we extend the standard definition of modal (Kripke) frames and models (Blackburn et al., 2001) for the *graded* modal operators from $\triangle$ by employing the confidence function μ and focussing on the minimal definition for ϕ. A *frame* $\mathcal{F}$ for the probabilistic modal language is a pair

$$\mathcal{F} = \langle \mathcal{W}, \triangle \rangle$$

where $\mathcal{W}$ is a non-empty set of *worlds* (or *situations, states, points, vertices, etc.*) and $\triangle$ a family of binary relations over $\mathcal{W} \times \mathcal{W}$, called *accessibility relations*. In the following, we *overload* $\delta \in \triangle$ below in that we let δ both refer to the modal in the syntax as well as to the accessibility relation R_δ in the semantics.

A *model* $\mathcal{M}$ for the probabilistic modal language is a triple

$$\mathcal{M} = \langle \mathcal{F}, \mathcal{V}, \mu \rangle$$

such that $\mathcal{F}$ is a *frame*, $\mathcal{V}$ is a *valuation*, assigning each proposition ϕ a subset of $\mathcal{W}$, viz., the set of worlds in which ϕ holds, and μ is a mapping, returning the confidence interval for a given modality from $\triangle$. Note that we only require a definition for μ in $\mathcal{M}$ (the model, but *not* in the frame), as $\mathcal{F}$ represents the relational structure without interpreting the edge labeling (the modal names) of the graph.

The *satisfaction relation* $\models$, given a model $\mathcal{M}$ and a specific world w is inductively defined over the set of well-formed formulae in *negation normal form* (remember $\pi ::= p(\vec{c}) \mid \neg p(\vec{c})$):

1. $\mathcal{M}, w \models p(\vec{c})$ **iff** $w \in \mathcal{V}(p(\vec{c}))$ **and** $w \notin \mathcal{V}(\neg p(\vec{c}))$
2. $\mathcal{M}, w \models \neg p(\vec{c})$ **iff** $w \in \mathcal{V}(\neg p(\vec{c}))$ **and** $w \notin \mathcal{V}(p(\vec{c}))$
3. $\mathcal{M}, w \models \phi \wedge \phi'$ **iff** $\mathcal{M}, w \models \phi$ **and** $\mathcal{M}, w \models \phi'$
4. $\mathcal{M}, w \models \phi \vee \phi'$ **iff** $\mathcal{M}, w \models \phi$ **or** $\mathcal{M}, w \models \phi'$
5. **for all** $\delta \in \triangle$: $\mathcal{M}, w \models \delta\pi$ **iff**

$$\frac{\#\{u \mid (w,u) \in \delta \textbf{ and } \mathcal{M}, u \models \pi\}}{\# \cup_{\delta' \in \triangle} \{u \mid (w,u) \in \delta'\}} \in \mu(\delta)$$

The last case of the satisfaction relation addresses the modals: for a world w, we look for the successor states u that are directly reachable via δ and in which π holds, and divide the number of such states ($\# \cdot$) by the number of all worlds that are reachable from w in the denominator. This number, lying between 0 and 1, is then required to be an element of the confidence interval $\mu(\delta)$ of δ in order to satisfy $\delta\pi$, given $\mathcal{M}, w$.

It is worth noting that the satisfaction relation above differs from the standard definition in its handling of $\mathcal{M}, w \models \neg p(\vec{c})$, as negation is *not* interpreted through the *absence* of $p(\vec{c})$ ($\mathcal{M}, w \not\models p(\vec{c})$), but through the *existence* of $\neg p(\vec{c})$. This treatment addresses the *open-world* nature in OWL and the evolvement of a (medical) domain over time.

We also note that the definition of the satisfaction relation for modalities (last clause) is related to the *possibility operators* $M_k \cdot$ (= $\Diamond^{\geq k} \cdot$; $k \in \mathbb{N}$) introduced by (Fine, 1972) and *counting modalities* $\cdot \geq n$ (Areces et al., 2010), used in modal logic characterizations of *description logics* with *cardinality* restrictions.

4.3 Well-Behaved Frames

As we will see later, it is handy to assume that the graded modals are arranged in a kind of hierarchy—the more we move along the arrows in the hierarchy, the more a statement ϕ in the scope of a modal $\delta \in \triangle$ becomes *uncertain*. In order to address this, we slightly extend the notion of a *frame* by a third component $\preceq \subseteq \triangle \times \triangle$, a partial order (i.e., a reflexive, antisymmetric, and transitive binary relation) between modalities:

$$\mathcal{F} = \langle \mathcal{W}, \triangle, \preceq \rangle$$

Let us consider the following modal hierarchy that we build from the set $\triangle$ of already introduced modals (cf. Figure 1):

$$\begin{tikzcd}
 & \top \arrow[r] & C \arrow[r] & L \arrow[r] & N \arrow[dr] & \\
! \arrow[ur] \arrow[dr] & & & & & ? \\
 & \bot \arrow[r] & E \arrow[r] & U \arrow[urrr] & &
\end{tikzcd}$$

This graphical representation is just a compact way to specify a set of 33 binary relation instances over $\triangle \times \triangle$, such as $\top \preceq \top$, $\top \preceq N$, $C \preceq N$, $\bot \preceq ?$, or $! \preceq ?$. The above mentioned form of uncertainty is expressed by the measure function μ in that the associated confidence intervals become larger:

$$\textbf{if } \delta \preceq \delta' \textbf{ then } \mu(\delta) \subseteq \mu(\delta')$$

In order to arrive at a proper and intuitive model-theoretic semantics which mirrors intuitions such as **if** ϕ *is confirmed* ($C\phi$) **then** ϕ *is likely* ($L\phi$), we will focus here on *well-behaved* frames $\mathcal{F}$ which enforce the existence of edges in $\mathcal{W}$, given $\preceq$ and $\delta, \delta^{\uparrow} \in \triangle$:

$$\textbf{if } (w,u) \in \delta \textbf{ and } \delta \preceq \delta^{\uparrow} \textbf{ then } (w,u) \in \delta^{\uparrow}$$

However, by imposing this constraint, we also need to adapt the last case of the satisfiability relation from Section 4.2 above:

5. **for all** $\delta \in \triangle$: $\mathcal{M}, w \models \delta\pi$ **iff**

$$\frac{\# \cup_{\delta^{\uparrow} \succeq \delta} \{u \mid (w,u) \in \delta^{\uparrow} \textbf{ and } \mathcal{M}, u \models \pi\}}{\# \cup_{\delta' \in \triangle} \{u \mid (w,u) \in \delta'\}} \in \mu(\delta)$$

Not only are we scanning for edges (w,u) labeled with δ and for successor states u of w in which π holds in the numerator (original definition), but also take into account edges marked with more general modals $\delta^{\uparrow}$: $\delta^{\uparrow} \succeq \delta$. This mechanism implements a kind of *built-in model completion* that is not necessary in ordinary modal logics as they deal with only a *single* relation (viz., unlabeled arcs).

5 ENTAILMENT RULES

We now turn our attention, again, to the syntax of our language and to the syntactic consequence relation. This section addresses a restricted subset of entailment rules which will unveil new (or implicit) knowledge from already existing graded statements. Recall that these kind of statements (in negation normal form) are a consequence of the application of simplification rules as depicted in Section 4.1. Thus, we assume a *pre-processing step* here that "massages" more complex statements that arise from a representation of graded (medical) statements in *natural language*. The entailments which we will present in a moment can either be *directly* implemented in a *tuple*-based reasoner, such as *HFC* (Krieger, 2013), or in *triple*-based engines (e.g., *Jena* (Carroll et al., 2004) or *OWLIM* (Bishop et al., 2011)) which need to *reify* the medical statements in order to be compliant with the RDF triple model.

5.1 Modal Entailments

The entailments presented in this section deal with *plausible* inference centered around modals $\delta, \delta' \in \triangle$ which are, in part, also addressed in (Schulz et al., 2014) in a pure OWL setting. We use the implication sign $\rightarrow$ to depict the entailment rules

$$lhs \rightarrow rhs$$

which act as *completion* (or *materialization*) rules the way as described in, e.g., (Hayes, 2004) and (ter Horst, 2005), and used in today's *semantic repositories* (e.g., *OWLIM*). We sometimes even use the biconditional $\leftrightarrow$ to address that the LHS and the RHS are semantically equivalent, but will indicate the direction that should be used in a practical setting. As before, we define

$$\pi ::= p(\vec{c}) \mid \neg p(\vec{c})$$

We furthermore assume that for every modal $\delta \in \triangle$, a *complement* modal δ^{C} and a *mirror* modal δ^{M} exist (cf. Section 4.1).

5.1.1 Lift

$$(\mathsf{L}) \quad \pi \leftrightarrow \top\pi$$

This rule interprets propositional statements as special modal formulae. It might be dropped and can be seen as a pre-processing step. We have used it in the Hepatitis example above. Usage: left-to-right direction.

5.1.2 Generalize

$$(\mathsf{G}) \quad \delta\pi \wedge \delta \preceq \delta' \rightarrow \delta'\pi$$

This rule schema can be instantiated in various ways, using the modal hierarchy from Section 4.3, e.g., $\top\pi \rightarrow C\pi$, $C\pi \rightarrow L\pi$, or $E\pi \rightarrow U\pi$. It has been used in the Hepatitis example.

5.1.3 Complement

$$(\mathsf{C}) \quad \neg\delta\pi \leftrightarrow \delta^{\mathsf{C}}\pi$$

In principle, (C) is not needed in case the statement is already in negation normal form. This schema might be useful for natural language paraphrasing (explanation). Given $\triangle$, there are four possible instantiations: $E\pi \leftrightarrow \neg N\pi$, $N\pi \leftrightarrow \neg E\pi$, $?\pi \leftrightarrow \neg !\pi$, and $!\pi \leftrightarrow \neg ?\pi$.

5.1.4 Mirror

$$(\mathsf{M}) \quad \delta\neg\pi \leftrightarrow \delta^{\mathsf{M}}\pi$$

Again, (D) is in principle not needed as long as the modal proposition is in negation normal form, since we do allow for negated propositional statements $\neg p(\vec{c})$. This schema might be useful for natural language paraphrasing (explanation). For $\triangle$, there are six possible instantiations, viz., $E\pi \leftrightarrow C\neg\pi$, $C\pi \leftrightarrow E\neg\pi$, $L\pi \leftrightarrow U\neg\pi$, $U\pi \leftrightarrow L\neg\pi$, $\top\pi \leftrightarrow \bot\neg\pi$, and $\bot\pi \leftrightarrow \top\neg\pi$.

5.1.5 Uncertainty

$$\text{(U)} \quad \delta\pi \wedge \neg\delta\pi \leftrightarrow \delta\pi \wedge \delta^{\mathsf{C}}\pi \leftrightarrow ?\pi$$

The *co-occurrence* of $\delta\pi$ and $\neg\delta\pi$ does *not* imply logical *inconsistency* (propositional case: $\pi \wedge \neg\pi$), but leads to complete *uncertainty* about the validity of π. Remember that $\mu(?) = \mu(\delta) \uplus \mu(\delta^{\mathsf{C}}) = [0,1]$:

$$\begin{array}{l} \qquad 0 \qquad\qquad\qquad\quad 1 \\ \mu : |\text{—}\delta^{\mathsf{C}}\text{—}|\text{——}\delta\text{——}| \\ \qquad\quad \pi \qquad\qquad \pi \end{array}$$

Usage: left-to-right direction.

5.1.6 Negation

$$\text{(N)} \quad \delta(\pi \wedge \neg\pi) \leftrightarrow \delta\pi \wedge \delta\neg\pi \leftrightarrow \delta\pi \wedge \delta^{\mathsf{M}}\pi \leftrightarrow \delta^{\mathsf{M}}\neg\pi \wedge \delta^{\mathsf{M}}\pi \leftrightarrow \delta^{\mathsf{M}}(\pi \wedge \neg\pi)$$

(N) shows that $\delta(\pi \wedge \neg\pi)$ can be formulated equivalently by using the mirror modal δ^{M}:

$$\begin{array}{l} \qquad 0 \qquad\qquad\qquad\qquad\quad 1 \\ \mu : |\text{—}\delta^{\mathsf{M}}\text{—}|\text{———}|\text{—}\delta\text{—}| \\ \qquad \pi \wedge \neg\pi \qquad\quad \pi \wedge \neg\pi \end{array}$$

In general, (N) is *not* the modal counterpart of the *law of non-contradiction*, as $\pi \wedge \neg\pi$ is usually afflicted by uncertainty, meaning that from $\delta(\pi \wedge \neg\pi)$, we can *not* infer that $\pi \wedge \neg\pi$ is the case for the concrete example in question (recall the intention behind the confidence intervals; cf. Section 3). There is one notable exception, involving the $\top$ and $\bot$ modals. This is formulated by the next entailment rule.

5.1.7 Error

$$\text{(E)} \quad \top(\pi \wedge \neg\pi) \leftrightarrow \bot(\pi \wedge \neg\pi) \rightarrow\ !(\pi \wedge \neg\pi) \leftrightarrow\ !\pi$$

(E) *is* the modal counterpart of the *law of non-contradiction* (note: $\bot^{\mathsf{M}} = \top, \top^{\mathsf{M}} = \bot, !^{\mathsf{M}} = !$). For this reason and *by definition*, the *error* (or *failure*) modal ! from Section 3 comes into play here. The modal ! can serve as a hint to either stop a computation the first time it occurs, or to continue reasoning and to syntactically memorize the ground literal π. Usage: left-to-right direction.

5.2 Subsumption Entailments

As before, we define two subsets of $\triangle$, called $\underline{1} = \{\top, C, L, N\}$ and $\underline{0} = \{\bot, E, U\}$, thus $\underline{1}$ and $\underline{0}$ effectively become

$$\underline{1} = \{\top, C, L, N, U^{\mathsf{C}}\} \qquad \underline{0} = \{\bot, U, E, C^{\mathsf{C}}, L^{\mathsf{C}}, N^{\mathsf{M}}\}$$

due to the use of complement modals δ^{C} and mirror modals δ^{M} for every base modal $\delta \in \triangle$ and by assuming that $E = N^{\mathsf{C}}$, $E = C^{\mathsf{M}}$, $U = L^{\mathsf{M}}$, and $\bot = \top^{\mathsf{M}}$, together with the four "opposite" cases.

Now, let $\sqsubseteq$ abbreviate relation subsumption as known from description logics and realized through `rdfs:subClassOf` and `rdfs:subPropertyOf`. Given this, we define two further very practical and plausible modal entailments which can be seen as the modal extension of the entailment rules (rdfs9) and (rdfs7) for classes and properties in RDFS (Hayes, 2004):

$$\text{(S1)} \quad \underline{1}p(\vec{c}) \wedge p \sqsubseteq q \rightarrow \underline{1}q(\vec{c})$$
$$\text{(S0)} \quad \underline{0}q(\vec{c}) \wedge p \sqsubseteq q \rightarrow \underline{0}p(\vec{c})$$

Note how the use of p and q switches in the antecedent and the consequent, even though $p \sqsubseteq q$ holds in both cases. Note further that propositional statements π are restricted to the positive case $p(\vec{c})$ and $q(\vec{c})$, as their negation in the antecedent will not lead to any valid entailments.

Here are four *instantiations* of (S0) and (S1) for the unary and binary case (remember, $C \in \underline{1}$ and $E \in \underline{0}$):

$\mathsf{ViralHepatitisB} \sqsubseteq \mathsf{ViralHepatitis} \wedge$
$C\mathsf{ViralHepatitisB}(x) \rightarrow C\mathsf{ViralHepatitis}(x)$

$\mathsf{ViralHepatitis} \sqsubseteq \mathsf{Hepatitis} \wedge$
$E\mathsf{Hepatitis}(x) \rightarrow E\mathsf{ViralHepatitis}(x)$

$\mathsf{deeplyEnclosedIn} \sqsubseteq \mathsf{containedIn} \wedge$
$C\mathsf{deeplyEnclosedIn}(x,y) \rightarrow C\mathsf{containedIn}(x,y)$

$\mathsf{superficiallyLocatedIn} \sqsubseteq \mathsf{containedIn} \wedge$
$E\mathsf{containedIn}(x,y) \rightarrow E\mathsf{superficiallyLocatedIn}(x,y)$

5.3 Extended RDFS & OWL Entailments

In this section, we will consider further entailment rules for RDFS (Hayes, 2004) and a restricted subset of OWL (ter Horst, 2005; Motik et al., 2012). Remember that modals only head positive and negative propositional letters π, not TBox or RBox axioms. Concerning the original entailment rules, we will distinguish *four principal cases* to which the extended rules belong (we will only consider the unary and binary case here as used in description logics/OWL):

1. TBox and RBox axiom schemas will not undergo a modal extension;
2. rules get extended in the antecedent;
3. rules take over modals from the antecedent to the consequent;
4. rules aggregate several modals from the antecedent in the consequent.

We will illustrate the individual cases in the following subsections with examples by using a kind of description logic rule syntax. Clearly, the set of extended entailments depicted here is *not complete*.

5.3.1 Case-1: No Modals

Entailment rule (rdfs11) from (Hayes, 2004) deals with class subsumption: $\mathsf{C} \sqsubseteq \mathsf{D} \wedge \mathsf{D} \sqsubseteq \mathsf{E} \rightarrow \mathsf{C} \sqsubseteq \mathsf{E}$. As this is a terminological axiom schema, the rule stays *constant* in the modal domain. Example rule instantiation:

$\mathsf{ViralHepatitisB} \sqsubseteq \mathsf{ViralHepatitis} \wedge$
$\mathsf{ViralHepatitis} \sqsubseteq \mathsf{Hepatitis} \rightarrow$
$\mathsf{ViralHepatitisB} \sqsubseteq \mathsf{Hepatitis}$

5.3.2 Case-2: Modals on LHS, No Modals on RHS

The following original rule (rdfs3) from (Hayes, 2004) imposes a range restriction on objects of binary ABox relation instances: $\forall \mathsf{P.C} \wedge \mathsf{P(x,y)} \rightarrow \mathsf{C(y)}$. The extended version needs to address the ABox proposition in the antecedent (*don't care* modal δ), but must not change the consequent (even though we always use the $\top$ modality here—the range restriction C(y) is always true, independent of the uncertainty of P(x, y); cf. Section 2 example):

$$\text{(Mrdfs3)} \quad \forall \mathsf{P.C} \wedge \delta\mathsf{P(x,y)} \rightarrow \top\mathsf{C(y)}$$

Example rule instantiation:

$\forall \mathsf{sufferesFrom.Disease} \wedge L\mathsf{suffersFrom}(x,y) \rightarrow$
$\top\mathsf{Disease}(y)$

5.3.3 Case-3: Keeping LHS Modals on RHS

Inverse properties switch their arguments (ter Horst, 2005) as described by (rdfp8): $\mathsf{P} \equiv \mathsf{Q}^- \wedge \mathsf{P(x,y)} \rightarrow \mathsf{Q(y,x)}$. The extended version simply keeps the modal operator:

$$\text{(Mrdfp8)} \quad \mathsf{P} \equiv \mathsf{Q}^- \wedge \delta\mathsf{P(x,y)} \rightarrow \delta\mathsf{Q(y,x)}$$

Example rule instantiation:

$\mathsf{containedIn} \equiv \mathsf{contains}^- \wedge C\mathsf{containedIn}(x,y) \rightarrow$
$C\mathsf{contains}(y,x)$

5.3.4 Case-4: Aggregating LHS Modals on RHS

Now comes the most interesting case of modalized RDFS & OWL entailment rules, that offers several possibilities on a varying scale between *skeptical* and *credulous* entailments, depending on the degree of uncertainty, as expressed by the measuring function μ of the modal operator. Consider the original rule (rdfp4) from (ter Horst, 2005) for transitive properties:

$\mathsf{P}^+ \sqsubseteq \mathsf{P} \wedge \mathsf{P(x,y)} \wedge \mathsf{P(y,z)} \rightarrow \mathsf{P(x,z)}$.

Now, how does the modal on the RHS of the extended rule look like, depending on the two LHS modals? There are several possibilities. By operating directly on the *modal hierarchy*, we are allowed to talk about, e.g., the *least upper bound* or the *greatest lower bound* of δ' and δ''. When taking the associated *confidence intervals* into account, we might play with the low and high numbers of the intervals, say, by applying min/max, the *arithmetic mean* or even by *multiplying* the corresponding numbers.

Let us first consider the general rule from which more specialized versions can be derived, simply by instantiating the combination operator $\odot$:

$$\text{(Mrdfp4)} \quad \mathsf{P}^+ \sqsubseteq \mathsf{P} \wedge \delta'\mathsf{P(x,y)} \wedge \delta''\mathsf{P(y,z)} \rightarrow (\delta' \odot \delta'')\mathsf{P(x,z)}$$

Here is an instantiation of Mrdfp4, dealing with the transitive relation contains from above, assuming that $\odot$ reduces to the *least upper bound* (i.e., $C \odot L = L$):

$C\mathsf{contains}(x,y) \wedge L\mathsf{contains}(y,z) \rightarrow$
$L\mathsf{contains}(x,z)$

What is the general result of $\delta' \odot \delta''$? It depends, probably both on the application domain and the *epistemic commitment* one is willing to accept about the "meaning" of gradation words/modal operators. To enforce that $\odot$ is at least both *commutative* and *associative* (as is the least upper bound) is probably a good idea, making the sequence of modal clauses *order independent*. And to work on the modal hierarchy instead of combining low/high numbers of the corresponding intervals is probably a good decision for forward chaining engines, as the latter strategy might introduce *new* individuals through operations such as multiplication, thus posing a problem for the implementation of the generalization schema (G) (see Section 5.1.2).

5.4 Custom Entailments: An Example from the Medical Domain

Consider that Hepatitis B is an infectious disease

$\mathsf{ViralHepatitisB} \sqsubseteq \mathsf{InfectiousDisease} \sqsubseteq \mathsf{Disease}$

and note that there exist vaccines against it. Assume that the liver l of patient p quite hurts

$C\mathsf{hasPain(p, l)}$,

but p has been definitely vaccinated against Hepatitis B before:

$\top\mathsf{vaccinatedAgainst(p, ViralHepatitisB)}$.

We apply OWL2-like punning here when using the class ViralHepatitisB (*not* an instance), as the second argument of vaccinatedAgainst; cf. (Golbreich and Wallace, 2012).

Given that p received a vaccination, the following custom rule will *not* fire (x, y below are now universally-quantified variables; z an existentially-quantified RHS-only variable):

$\top\mathsf{Patient}(x) \wedge \top\mathsf{Liver}(y) \wedge C\mathsf{hasPain}(x,y) \wedge$
$U\mathsf{vaccinatedAgainst}(x, \mathsf{ViralHepatitisB}) \rightarrow$
$N\mathsf{ViralHepatitisB}(z) \wedge N\mathsf{suffersFrom}(x,z)$

Now assume another person p' that is pretty sure (s)he was never vaccinated:

$E\mathsf{vaccinatedAgainst}(\mathsf{p}', \mathsf{ViralHepatitisB})$

Given the above custom rule, we are allowed to infer that (h instantiation of z)

$N\mathsf{ViralHepatitisB}(\mathsf{h}) \wedge N\mathsf{suffersFrom}(\mathsf{p}', \mathsf{h})$

The subclass axiom from above thus assigns

$N\mathsf{InfectiousDisease}(\mathsf{h})$

so that we can query for patients for whom an infectious disease is *not excluded* (= N), in order to initiate appropriate methods (e.g., further medical investigations).

5.5 Implementing Modal Entailments

The negation normal form from Section 4.1 makes it relatively easy to implement entailment rules involving modalized propositional letters of the form $\delta \pm p(\vec{c})$. $\pm$ is a *polarity value* as known from situation theory (Devlin, 2006) in order to make negative property assertions available in the object language.

We have implemented a modalized extension of the RDFS and OWL rule sets (Hayes, 2004; ter Horst, 2005) by employing the tuple-based rule engine *HFC* (Krieger, 2012; Krieger, 2013). Without loss of generality, let us focus here on the positive case for the three binary entailment schemas from Section 5.3.2, 5.3.3, and 5.3.4 and their *HFC* rule representation, as negation inside the scope of a modal can be rewritten using the mirror modal, thus turning the quintuple into a quad (rule variables start with a `?`):

(Mrdfs3) $\forall \mathsf{P.C} \wedge \delta\mathsf{P(x,y)} \rightarrow \top\mathsf{C(y)}$

```
?p rdfs:range ?c
?modal ?x ?p ?y
->
mod:T ?y rdf:type ?c
```

(Mrdfp8) $\mathsf{P} \equiv \mathsf{Q}^- \wedge \delta\mathsf{P(x,y)} \rightarrow \delta\mathsf{Q(y,x)}$

```
?p owl:inverseOf ?q
?modal ?x ?p ?y
->
?modal ?y ?q ?x
```

(Mrdfp4) $\mathsf{P}^+ \sqsubseteq \mathsf{P} \wedge \delta\mathsf{P(x,y)} \wedge \delta'\mathsf{P(y,z)} \rightarrow (\delta \odot \delta')\mathsf{P(x,z)}$

```
?p rdf:type owl:TransitiveProperty
?modal1 ?x ?p ?y
?modal2 ?y ?p ?z
->
?modal ?x ?p ?z
@action
?modal = CombineModals ?modal1 ?modal2
```

Triple-based engines, such as *OWLIM* clearly need to reify such extended descriptions (expensive; no termination guarantee). Even more important, additional tests going beyond simple symbol matching and function calls, such as `CombineModals` (the equivalent to $\odot$ in the abstract syntax) in the *HFC* version of (Mrdfp4) above, are rarely available in today's RDFS/OWL reasoning engines, thus making it impossible for them to implement such modal entailments.

We finally describe how the implementation of the generalization schema (G) (Section 5.1.2) works. As explained in Section 4.3, the modal operators δ are arranged in a modal hierarchy that is based on the inclusion of their confidence intervals $\mu(\delta)$. This hierarchy is realized in OWL through a subclass hierarchy, using `rdfs:subClassOf` to implement $\preceq$:

(G) $\delta\mathsf{P(x,y)} \wedge \delta \preceq \delta' \rightarrow \delta'\mathsf{P(x,y)}$

```
?modal1 ?x ?p ?y
?modal1 rdfs:subClassOf ?modal2
->
?modal2 ?x ?p ?z
```

6 A FOURTH KIND OF MODALS

The two modalities $\Box$ and $\Diamond$ from standard modal logic are often called *dual* as they can be defined in terms of each other: $\Box\phi \equiv \neg\Diamond\neg\phi$ and $\Diamond\phi \equiv \neg\Box\neg\phi$, resp. At first sight, it seems that our non-standard modal logic is missing a similar property, as we originally dealt with *five* modal operators, extended by the *propositional* modals $\top$ and $\bot$, and the *completion* modals ? and !. For every such modal δ, we can furthermore think of additional *complement* modals δ^{C} and additional *mirror* modals δ^{M} whose confidence intervals $\mu(\delta^{\mathsf{C}})$ and $\mu(\delta^{\mathsf{M}})$ can be derived from $\mu(\delta)$ (cf. Section 4.1). Some of these modals coincide with original modals from $\triangle$, others do not have a direct counterpart. However, the confidence intervals for the "anonymous" modals can be trivially computed by applying the two equations from Section 4.1.

Coming back to the question of whether dual modals exist for every $\delta \in \triangle$, we need to *simplify* $\neg\delta\neg\phi$ by applying the schemas from Section 4.1. We can either start with the inner or with the outer negation, resulting in either mirror modals or complement modals. Interestingly, the resulting confidence intervals at which we reach in the end are the *same*, and this is clearly a good point and desirable, as simplification is supposed to be an *order-independent* process:

$$\begin{array}{ccc} & \neg\delta\neg\phi & \\ / & & \backslash \\ \delta^{\mathsf{C}}\neg\phi & & \neg\delta^{\mathsf{M}}\phi \\ | & & | \\ \delta^{\mathsf{C}^{\mathsf{M}}}\phi & & \delta^{\mathsf{M}^{\mathsf{C}}}\phi \end{array}$$

Thus, $\delta^{\mathsf{C}^{\mathsf{M}}} \equiv \delta^{\mathsf{M}^{\mathsf{C}}}$, for every $\delta \in \triangle$ which can be shown by applying the definitions for complement and mirror modals from Section 4.1. The deeper reason why this is so is related to the inherent properties of the two operations *complementation* and *mirroring*. Contrary to complement and mirror modals, dual modals δ^{D} are either supersets or subsets of $\mu(\delta)$, i.e., if δ is a $\underline{1}$- or $\underline{0}$-modal, so is δ^{D}.

7 RELATED WORK & REMARKS

It is worth noting to state that this paper is interested in the representation of and reasoning with *uncertain assertional* knowledge, and neither in dealing with *vagueness/fuzziness* found in natural language (*very small, hot*), nor in handling *defaults* and *exceptions* in *terminological* knowledge (*penguins can't fly*).

To the best of our knowledge, the modal logic presented in this paper uses for the first time modal operators for expressing the degree of (un)certainty of propositions. These modal operators are interpreted in the model theory through confidence intervals via measure function μ. From a model point of view, our modal operators are related to *counting modalities* $\Diamond^{\geq k}$ (Fine, 1972; Areces et al., 2010). However, for $\mathcal{M}, w \models \delta\pi$ to be the case, we do *not* require a *fixed* number $k \in \mathbb{N}$ of reachable successor states (*absolute* frequency), but instead *divide* the number of worlds reached through label $\delta \in \triangle$ and in which π holds by the number of all directly reachable worlds, yielding fraction $0 \leq p \leq 1$. This number then is further constrained by requiring $p \in \mu(\delta)$ (*relative* frequency), as defined in case 5 of the satisfaction relation in Section 4.2 and extended in Section 4.3.

As (Wikipedia, 2015) precisely put it: "... *what axioms and rules must be added to the propositional calculus to create a usable system of modal logic is a matter of philosophical opinion, often driven by the theorems one wishes to prove* ...". Clearly, the logic presented here is *no* exception and its design is driven by commonsense knowledge and plausible inferences, we try to capture and generalize. In a *strict* sense, it is a non-standard modal logic in that it is *not* an instance of the *normal* modal logic $\mathbf{K} = (N) + (K)$

$(N)\ p \rightarrow \Box p$
$(K)\ \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$

as the *necessitation rule* (N) and the *distribution axiom* (K) does *not* hold for *every* $\delta \in \triangle$. However, we can show that restricted generalized forms of these axioms are in fact the case for our logic ($\underline{1}^{\geq 0.5}$ are $\underline{1}$-modals whose low value is ≥ 0.5 and $\underline{0}^{\leq 0.5}$ are $\underline{0}$-modals whose high value is ≤ 0.5):

$(N1)\ p \rightarrow \underline{1}p$
$(N0)\ \neg p \rightarrow \underline{0}p$
$(K1^{\geq 0.5})\ \underline{1}^{\geq 0.5}(p \rightarrow q) \rightarrow (\underline{1}^{\geq 0.5}p \rightarrow \underline{1}^{\geq 0.5}q)$
$(K0^{\leq 0.5})\ \underline{0}^{\leq 0.5}(p \rightarrow q) \rightarrow (\underline{0}^{\leq 0.5}p \rightarrow \underline{0}^{\leq 0.5}q)$

In addition, the *well-behaved frames* condition (Section 4.3) generalizes the *seriality* condition (D) on frames and a kind of *forward monotonicity*, we would like to keep for an evolving domain, is directly related to *transitivity* (4) of the accessibility relations from $\triangle$ in $\mathcal{F}$:

$(D)\ \delta p \wedge \delta \preceq \delta' \rightarrow \delta' p$
$(4)\ \delta p \rightarrow \delta\delta p$

Several approaches to *representing and reasoning with uncertainty* have been investigated in *Artificial Intelligence*; see (Halpern, 2003) for a (biased) overview. (Halpern, 1990) was probably the first attempt of a first-order logic which unifies probability distributions over classes *and* individuals. Weaker decidable propositional formalisms such as Bayesian Networks (Pearl, 1988) and related probabilistic graphical models (Koller and Friedmann, 2009) have found their way into causal (medical) reasoning (Lucas et al., 2004). Programming languages for these kind of models exist; e.g., Alchemy for *Markov Logic Networks* (Richardson and Domingos, 2006). In Markov Logic, first-order formulae are associated with a numerical value which *softens* hard first-order constraints and a violation makes a possible world not impossible, but less probable (the higher the weight, the stronger the rule). For example, the Markov Logic rule *smoking causes cancer* with weight 1.5 (Richardson and Domingos, 2006, p. 111)

$1.5 : \forall x \,.\, \mathit{smokes}(x) \rightarrow \mathit{hasCancer}(x)$

might be *approximated* in our approach through the use of modals:

$\top\mathsf{smokes(x)} \rightarrow L\mathsf{hasCancer(x)}$

Very less so has been researched in the *Description Logic* community (as it is smaller) and little or nothing of this research has find its way into implemented description logic systems. As we focus in this paper on a modalized extension of OWL, let us

$\triangle$	*meaning*	*confidence*	*belief*	*disbelief*	*uncertainty*
!	*error*	$\emptyset$	0.5	0.5	0
$\perp$	*false*	$[0,0]$	0	1	0
E	*excluded*	$[0,0.1]$	0	0.9	0.1
U	*unlikely*	$[0,0.3]$	0	0.7	0.3
PN	*perhaps not*	$[0.4,0.5]$	0.4	0.5	0.1
FF	*fifty-fifty*	$[0.45,0.55]$	0.45	0.45	0.1
P	*perhaps*	$[0.5,0.6]$	0.5	0.4	0.1
N	*not excluded*	$[0.1,1]$	0.3	0	0.7
L	*likely*	$[0.7,1]$	0.7	0	0.3
C	*confirmed*	$[0.9,1]$	0.9	0	0.1
$\top$	*true*	$[1,1]$	1	0	0
?	*unknown*	$[0,1]$	0	0	1

Figure 2: Representation of modal operators from $\triangle$ (incl. three *in-the-middle* modals) in terms of *opinions* in Subjective Logic. The confidence intervals for the five initial modals roughly coincide with the numbers depicted in Figure 1.

review here some of the work carried out in description logics. (Heinsohn, 1993) and (Jaeger, 1994) consider *uncertainty* in $\mathcal{ALC}$ concept hierarchies, plus concept typing of individuals (unary relations) in different ways (probability values vs. intervals; conditional probabilities in TBox vs. TBox+ABox). They do not address uncertain binary (or even n-ary) relations. (Tresp and Molitor, 1998) investigates *vagueness* in $\mathcal{ALC}$ concept descriptions to address statements, such as *the patient's temperature is high*, but also for determining membership degree ($\overline{38.5}$ $°C$). This is achieved through *membership manipulators* which are functions, returning a truth value between 0 and 1, thus deviating from a two-valued logic. (Straccia, 2001) defines a *fuzzy* extension of $\mathcal{ALC}$, based on Zadeh's *Fuzzy Logic*. As in (Tresp and Molitor, 1998), the truth value of an assertion is replaced by a membership value from $[0,1]$. $\mathcal{ALC}$ assertions α in (Straccia, 2001) are made fuzzy by writing, e.g., $\langle \alpha \geq n \rangle$, thus taking a single truth value from $[0,1]$. An even more expressive theoretical description logic, Fuzzy OWL, based on OWL DL, is investigated in (Stoilos et al., 2005).

Our work might also be viewed as a modalized version of a restricted fragment of *Subjective Logic* (Jøsang, 1997; Jøsang, 2001), a probabilistic logic that can be seen as an extension of Dempster-Shafer belief theory (Wilson, 2000). Subjective Logic addresses subjective believes by requiring numerical values for *believe b*, *disbelieve d*, and *uncertainty u*, called (subjective) opinions. For each proposition, it is required that $b+d+u=1$.

The translation from modals δ to $\langle b,d,u \rangle$ is determined by the length of the confidence interval $\mu(\delta) = [l,h]$ and its starting/ending numbers, viz., $u := h-l$, $b := l$, and $d := 1-h$ (cf. Figure 2).

These definitions also address *in-the-middle* modals (cf. footnote 3). Such modals even do *not* need to be symmetrical, i.e., being around the center of the confidence interval. The definitions are clearly not applicable to the error modal ! (cf. Section 5.1.7) and it makes perfect sense to assume $u = 0$ here (remember, $\mu(!) = \emptyset$), and thus bisecting the belief mass for this corner case, i.e., $b = 0.5$ and $d = 0.5$.

The simplification and entailment rules of the formalism (Sections 4.1 and 5) allow rule-based (forward) engines to easily implement this conservative extension of OWL. Through these rules, the formalism is *compositional* by nature and thus afflicted with all the problems, reviewers have already noted on the interplay between logic and uncertainty (Dubois and Prade, 1994). Due to the finite number of modal operators, the approach is only able to approximately compute the degree of uncertainty of new knowledge instead of giving more precise estimations, by combining the low/high numbers of the confidence intervals through min/max, multiplication, addition, etc. Contrary to other approaches, we do *not* talk about the uncertainty of *complex* propositions (conjunction, disjunction) or *sets* of beliefs, but instead focus merely on the uncertainty of *atomic* ABox propositions.

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