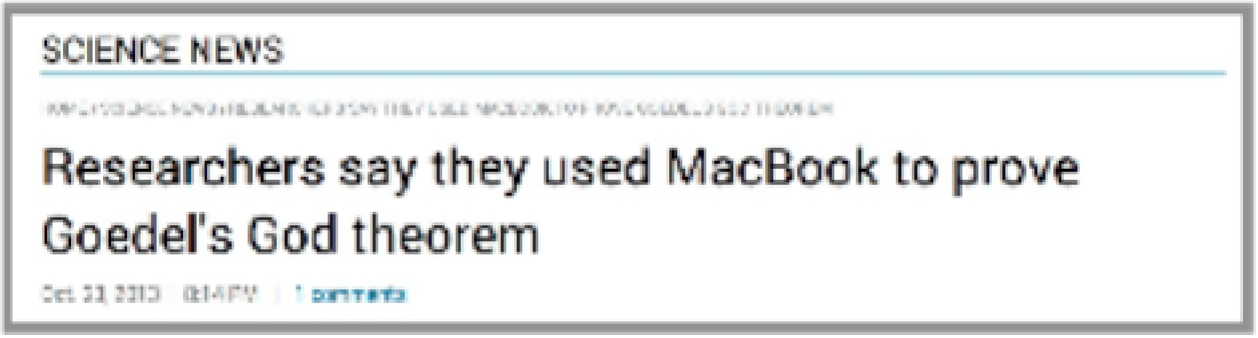
A
Proof
of
God

(Christoph Benzmüller and Bruno Woltzenlogel Paleo, 2013, 2015, 2016)

ISSUES

- **History of logic (a famous example)**
- **More expressive logics for NL representations**
- **Handling these logics for proving**
- **A glimpse on the state-of-the-art in logics**
- **Useful for modern philosophy**
- **Media interest (rare for technical issues)**

IN A NUTSHELL


SCIENCE NEWS

Researchers say they used MacBook to prove Goedel's God theorem

Techniques

- Advanced proof techniques required (higher order, modal logic)
- Progress in use of logics obtained

Media interest

- Germany: Spiegel Online, FAZ, Die Welt, …
- International: Austria, Italy, India, US, …

HISTORICAL BACKGROUND

Ontological argument

- **Deductive argument (for the existence of God)**
- **Starting from premises, justified by pure reasoning**

Rich history of ontological arguments

- **Pro: Descartes, Leibniz, Hegel, Gödel, …**
- **Against: Th. Aquinas, Kant, Frege, …**

Gödels notion of god

"A God-like being possesses all positive properties"

->

"(Necessarily) God exists"

proved by Gödel on two hand-written pages

Reasoning about Arguments

God is an entity of which nothing greater can be conceived (*Anselm*)
existence in the actual world would make
such an assumed being even greater

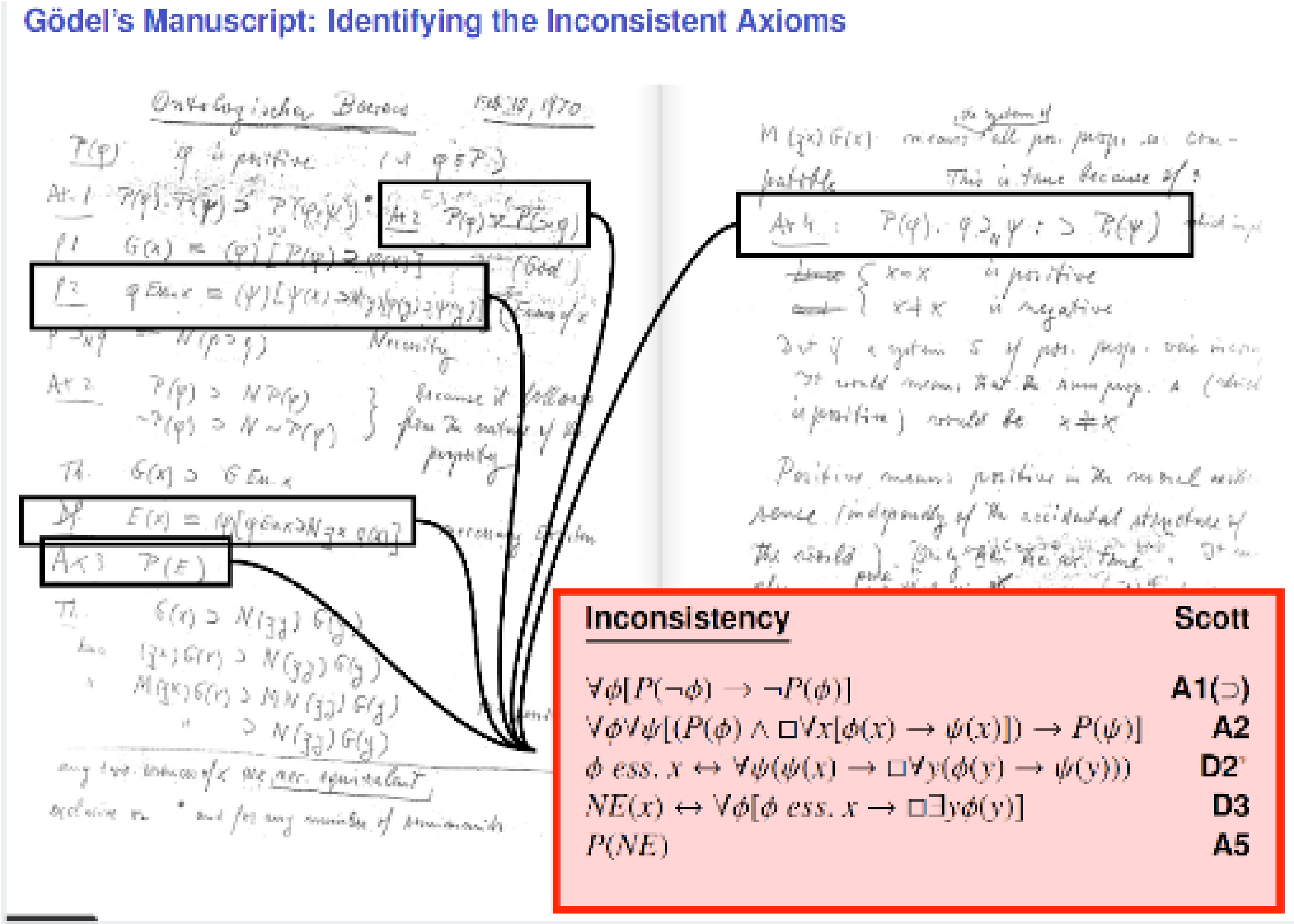
Leibniz: the assumption should be derivable from the definition of God

***Gödel* explicitly proves that God's existence is possible**

Gödels axioms considered too strong

Ongoing philosophical debate

GÖDEL'S HAND-WRITTEN PROOF



Scott's Version of Gödel's Axioms, Definitions and Theorems

Axiom A1 Either a property or its negation is positive, but not both: $\forall \phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

Axiom A2 A property necessarily implied by a positive property is positive: $\forall \phi \forall \psi[(P(\phi) \wedge \Box \forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

Thm. T1 Positive properties are possibly exemplified: $\forall \phi[P(\phi) \rightarrow \Diamond \exists x \phi(x)]$

Def. D1 A *God-like* being possesses all positive properties: $G(x) \leftrightarrow \forall \phi[P(\phi) \rightarrow \phi(x)]$

Axiom A3 The property of being God-like is positive: $P(G)$

Cor. C Possibly, God exists: $\Diamond \exists x G(x)$

Axiom A4 Positive properties are necessarily positive: $\forall \phi[P(\phi) \rightarrow \Box P(\phi)]$

Def. D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi\ ess.\ x \leftrightarrow \boxed{\phi(x) \wedge} \forall \psi(\psi(x) \rightarrow \Box \forall y(\phi(y) \rightarrow \psi(y)))$

Thm. T2 Being God-like is an essence of any God-like being: $\forall x[G(x) \rightarrow G\ ess.\ x]$

Def. D3 *Necessary existence* of an individual is the necessary exemplification of all its essences: $NE(x) \leftrightarrow \forall \phi[\phi\ ess.\ x \rightarrow \Box \exists y \phi(y)]$

Axiom A5 Necessary existence is a positive property: $P(NE)$

Thm. T3 Necessarily, God exists: $\Box \exists x G(x)$

Difference to Gödel (who omits this conjunct)

SCOTT'S VERSION OF GÖDEL'S

AXIOMS, DEFINITIONS AND THEOREMS

A1 Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

A2 A property necessarily implied by a positive property is positive: $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \to \psi(x)]) \to P(\psi)]$

T1 Positive properties are possibly exemplified: $\forall\varphi[P(\varphi) \to \Diamond\exists x\varphi(x)]$

D1 A *God-like* being possesses all positive properties: $G(x) \leftrightarrow \forall\phi[P(\phi) \to \phi(x)]$

A3 The property of being God-like is positive: $P(G)$

C Possibly, God exists: $\Diamond\exists x G(x)$

A4 Positive properties are necessarily positive: $\forall\phi[P(\phi) \to \Box\, P(\phi)]$

D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi \text{ } ess.\ x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \to \Box\forall y(\phi(y) \to \psi(y)))$

T2 Being God-like is an essence of any God-like being: $\forall x[G(x) \to G \text{ } ess.\ x]$

D3 *Necessary existence* of an individual is the necessary exemplification of all its essences: $NE(x) \leftrightarrow \forall\phi[\phi \text{ } ess.\ x \to \Box\exists y\phi(y)]$

A5 Necessary existence is a positive property: $P(NE)$

T3 Necessarily, God exists: $\Box\exists x G(x)$

Important results

Inconsistency in Gödel's axioms previously unknown
first found by a machine (2013)

Fully automated proof (of Scotts version) of God's existence
first by Leo-II in 2016 (2,5 sec)

Issues and tendencies

Proposing/discussing different axiomatizations

Making them provably accessible by simpler logics

Checking the validity of metaphysics arguments

Providing interfaces/logical frameworks

PROOF OVERVIEW

(IN NATURAL DEDUCTION STYLE)

$$\dfrac{\overset{\textbf{A3}}{P(G)} \qquad \dfrac{\overset{\textbf{A2}}{\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]} \qquad \overset{\textbf{A1a}}{\forall\varphi.[P(\neg\varphi) \to \neg P(\varphi)]}}{\textbf{T1: } \forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]}}{\textbf{C1: } \Diamond\exists z.G(z)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\textbf{A1b}}{\forall\varphi.[\neg P(\varphi) \to P(\neg\varphi)]} \qquad \overset{\textbf{A4}}{\forall\varphi.[P(\varphi) \to \Box P(\varphi)]}}{\textbf{T2: } \forall y.[G(y) \to G\ ess\ y]} \qquad \overset{\textbf{A5}}{P(NE)}}{\textbf{L1: } \exists z.G(z) \to \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \to \Diamond\Box\exists x.G(x)} \qquad \overset{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \to \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

ONE PROOF STEP IN DETAIL
(IN NATURAL DEDUCTION STYLE)

$$
\dfrac{\dfrac{\dfrac{\overset{\star_4}{G(\gamma)} \qquad \dfrac{\overset{\text{Theorem 2}}{\forall x.G(x) \to G\ ess\ x}}{G(\gamma) \to G\ ess\ \gamma}\,\forall_E}{G\ ess\ \gamma^{\dagger}} \qquad \dfrac{\dfrac{\dfrac{\overset{\text{Axiom 5}}{P(E)} \qquad \dfrac{\dfrac{\overset{\star_4}{G(\gamma)}}{\forall \varphi.P(\varphi) \to \varphi(\gamma)}}{P(E) \to E(\gamma)}\,\forall_E}{E(\gamma)^{\dagger}}}{\forall \varphi.\varphi\ ess\ \gamma \to \Box\exists x.\varphi(x)}}{G\ ess\ \gamma \to \Box\exists x.G(x)}\,\forall_E}{\Box\exists x.G(x)}}{\exists x.G(x) \to \Box\exists x.G(x)}\,\to_I^1
$$

2 *If the existence of a God-like being is possible, then it is necessary:*

$$\Diamond\exists x.G(x) \to \Box\exists x.G(x)$$

PROOF DESIGN

State-of-the-art

- **No prover for higher order modal logic exists**
- **Several (increasingly better and coordinated) provers for higher order logic exist (interactive and automated ones)**

Overall strategy

- **Embedding in higher order classical logic (based on experience with embedding first-order modal logic in higher order logic)**
- **Making use of higher-order logic theorem provers**
- **Interactive proof oriented on human-designed natural deduction proof**

Assessment

A fully automated proof may be possible in about 3 years (2013)

Benzmüller

EMBEDDING IN HIGHER-ORDER LOGIC

QML $\varphi, \psi \; ::= \; \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x\, \varphi \mid \exists x\, \varphi \mid \forall P\, \varphi$

HOL $s, t \; ::= \; C \mid x \mid \lambda x s \mid s\, t \mid \neg s \mid s \vee t \mid \forall x\, t$

QML in HOL: QML formulas φ are mapped to HOL predicates $\varphi_{\iota \to o}$

$$
\begin{array}{lcl}
\neg & = & \lambda\varphi_{\iota\to o}\lambda s_\iota \neg\varphi s \\
\wedge & = & \lambda\varphi_{\iota\to o}\lambda\psi_{\iota\to o}\lambda s_\iota (\varphi s \wedge \psi s) \\
\rightarrow & = & \lambda\varphi_{\iota\to o}\lambda\psi_{\iota\to o}\lambda s_\iota (\neg\varphi s \vee \psi s) \\
\Box & = & \lambda\varphi_{\iota\to o}\lambda s_\iota \forall u_\iota\, (\neg rsu \vee \varphi u) \\
\Diamond & = & \lambda\varphi_{\iota\to o}\lambda s_\iota \exists u_\iota\, (rsu \wedge \varphi u) \\
\forall & = & \lambda h_{\mu\to(\iota\to o)}\lambda s_\iota \forall d_\mu\, hds \\
\exists & = & \lambda h_{\mu\to(\iota\to o)}\lambda s_\iota \exists d_\mu\, hds \\
\forall & = & \lambda H_{(\mu\to(\iota\to o))\to(\iota\to o)}\lambda s_\iota \forall d_\mu\, Hds \\
 & & \\
\text{valid} & = & \lambda\varphi_{\iota\to o}\forall w_\iota \varphi w
\end{array}
\qquad \text{Ax}
$$

The equations in Ax are given as axioms to the HOL provers!

(Remark: Note that we are here dealing with constant domain quantification)

Example for embedding modal logic (operators)

$$
\begin{aligned}
\Box\forall x P x \quad &\equiv \quad \Box\Pi'(\lambda x.\lambda w.Pxw) \\
&\equiv \quad \Box((\lambda\Phi.\lambda w.\Pi(\lambda x.\Phi xw))(\lambda x.\lambda w.Pxw)) \\
&\equiv \quad \Box(\lambda w.\Pi(\lambda x.(\lambda x.\lambda w.Pxw)xw)) \\
&\equiv \quad \Box(\lambda w.\Pi(\lambda x.Pxw)) \\
&\equiv \quad (\lambda\varphi.\lambda w.\forall v.(Rwv \rightarrow \varphi v))(\lambda w.\Pi(\lambda x.Pxw) \\
&\equiv \quad (\lambda\varphi.\lambda w.\Pi(\lambda v.Rwv \rightarrow \varphi v))(\lambda w.\Pi(\lambda x.Px \\
&\equiv \quad (\lambda w.\Pi(\lambda v.Rwv \rightarrow (\lambda w.\Pi(\lambda x.Pxw))v)) \\
&\equiv \quad (\lambda w.\Pi(\lambda v.Rwv \rightarrow \Pi(\lambda x.Pxv))) \\
&\equiv \quad (\lambda w.\forall v.Rwv \rightarrow \forall x.Pxv) \\
&\equiv \quad (\lambda w.\forall vx.Rwv \rightarrow Pxv)
\end{aligned}
$$

Getting [] transformed, followed by simplifications

COURSE OF THE PROOF

Subproof	*Prover responsible*
Checking consistency	**Nitpick (model checker)**
Checking consistency Gödels original definition of D2	**LEO II (ATP)**
Proving T1 (positive properties ev. exemplified) is a theorem	**LEO II (ATP)**
Proving C (possibly, God exists) is a theorem	**LEO II (ATP)**
Proving T2 (being God-like is an essence …) is a theorem	**LEO II (ATP)**
Proving T3 (necessarily God exists) is a theorem	**LEO II (ATP)**
Proving C2 (necessarily God exists) is a theorem	**LEO II (ATP)**
Checking axioms are consistent	**Nitpick (model checker)**
Checking Gödels original axioms are inconsistent	**LEO II (ATP)**
Checking modal collaps	**LEO II (ATP)**
Checking "flawlessness of God"	**LEO II (ATP)**
Proving Monotheism	**TPS (ATP)**

Main Findings

The axioms and definitions in Scotts formulation are consistent

A simpler logic is sufficient to prove sub theorems

A modal logic is needed to prove the main theorems
(thus disproves some criticism on Gödel's formalization)

Only the main theorems is challenging for theorem provers

Gödel's original version is inconsistent (definition of *essence*)

Gödel's axioms imply modal collapse ($\phi \supset []\phi$)
contingent truth implies necessary truth
(can even be interpreted as an argument against free will);

Gödel's axioms imply monotheism

CRITICISM & OUTLOOK

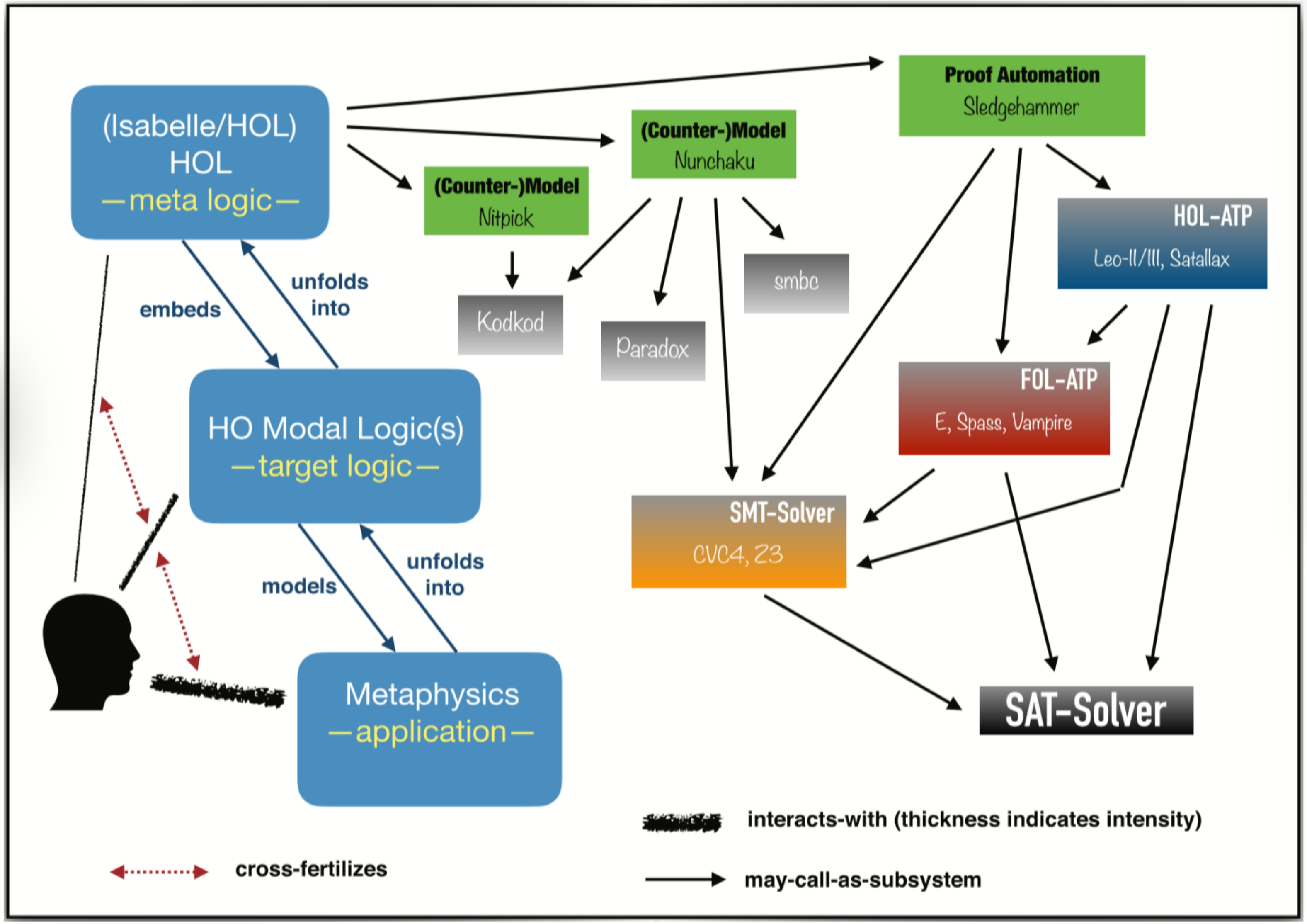
Problematic assumptions

- **Everything that is the case is so necessarily. $\forall P.[P \rightarrow \Box P]$**
  (follows from T2, T3, D2, proved by higher order ATPs)
  Then everything is determined, there is no free will …
- **Either a property or its negation is positive**
  in the morale sense, according to Gödel

Results

- **Powerful infrastucture to reason in higher-order modal logic**
- **Several insights about the strength of logics needed or not needed**
- **Difficult benchmark problems for higher-order theorem provers**
- **Major step towards computer-assisted theoretical philosophy**
- **Further ontological arguments to be tested (in particular, related to Gödel)**

(see http://page.mi.fu-berlin.de/cbenzmueller/, link presentations)


(Isabelle/HOL)
HOL
—meta logic—
(Counter-)Model
Nitpick
(Counter-)Model
Nunchaku
Proof Automation
Sledgehammer
HOL-ATP
Leo-II/III, Satallax
Kodkod
Paradox
smbc
FOL-ATP
E, Spass, Vampire
embeds
unfolds
into
HO Modal Logic(s)
—target logic—
SMT-Solver
CVC4, Z3
models
unfolds
into
Metaphysics
—application—
SAT-Solver
interacts-with (thickness indicates intensity)
cross-fertilizes
may-call-as-subsystem

STATE OF AFFAIRS OF THEOREM PROVERS

Capabilities

- **Occasional success with proofs of prominent theorems**
  (usually tedious and extremely longish, but first known formal result)
- **Some specialized provers**
  (taxonomic reasoners, equation provers)
- **Considerable progress in efficiency recently**

Variety of uses

- **Remote access to several ATPs (first-order, higher order)**
- **Calling several (distinct) provers in parallel (hoping at least one succeeds)**
- **Combining reasoning techniques (proving + computer algebra)**
- **Interactive proving (adding control for the prover, software verification)**
- **Proof planning – provers supported by proof schemas that encapsulate knowledge**