Modal Frame Characterization by Way of Auxiliary Modalities

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Abstract

In modal logics we are interested in classes of frames that characterize the logic under consideration. Such classes are usually distinguished by their respective frame properties. In general these characterizations are not unique and it is desirable to find a strongest possible. In this article an approach is presented which helps in this respect. It allows us to transform a given background theory into one which is more general and which modal logics cannot distinguish from the former because of their syntactic and semantic restrictions. The underlying technique is based on the idea to find conservative extensions of a given logic whose frame properties allow us to extract significantly stronger characterizations of the original logic.

1 Introduction

Normal modal logics are distinguished syntactically by axioms and inference rules and semantically by different classes of *frames* (a pair consisting of a set of *worlds* and a binary *accessibility relation* belonging to it). Latest since the early sixties it is known that there is a certain relationship between modal axioms and frame properties which somehow bridge the gap between syntax and semantics (see, e.g, [4]). Finding out about such relationships is nowadays a matter of the so-called *modal logic correspondence theory.*

A slightly different notion, the *modal logic completeness theory* – also related to the *modal logic model theory* – comes into play whenever we are interested in comparing accessibility relation properties with whole *logics* and not only with single axioms. In this case we say that a logic is *determined* or *characterized* by some theory (the accessibility relation properties) if and only if the theorems of the logic are exactly the formulas that are valid in all interpretations for which the theory holds. Interestingly, such theories are in general not unique, although the correspondence properties for modal axioms are. For instance it is known that the modal logic K is determined by the class of *all* frames, but also it is determined by the class of all *irreflexive* frames.[1] Similarly, although for some other reason, the modal logic $S5$ is determined by an equivalence relation but also by the universal relation. Again, modal logics are unable to tell the difference. Knowing about this is of quite some importance for several reasons: First, we learn about the limitations in the expressive power of modal languages, and second, reasoning within modal logics can be simplified in particular if it is based on semantic translations. Thus, whenever we are about to reason within modal logics on semantical grounds we have to be able to deal with the determining theory and, evidently, the simpler the theory is the easier the reasoning gets. We are therefore interested in finding theories that are as simple as possible. Usually this can be achieved if we have a means at hand with which we can extract a (in a sense) strongest such theory. Although it is not guaranteed that stronger theories are indeed simpler from a theorem prover's perspective they at least subsume the original one and therefore serve as a good candidate (witness the logic $S5$ where the universal relation is both stronger and simpler than the equivalence relation).

Quite a lot has been done in this field during the last decades. Typical notions that occur in the standard literature are *canonical frames*, *generated frames*, *filtration*, *bulldozing*, *unraveling*, *p-morphisms*, etc (see [1, 9, 10, 12]). All these techniques have in common that they construct for each frame an equivalent canonical (generated, filtration, unraveled, etc) frame with stronger properties and because of this equivalence modal logics cannot distinguish between them, hence the stronger properties may consistently be assumed.

In this paper a somewhat different technique is proposed for extracting such stronger properties. It also has to do with alternative frames, however, it is better characterized by the notion of *alternative axiomatizations.* The main idea is briefly sketched as follows: Given a modal logic L we first try to find an extension L^{ex} of L in a way such that all the newly derivable theorems do not belong to the language of L. Then we look for a determining property (characterization of the frames) for L^{ex} and that with

[1] It has become customary to speak of frames having a certain property when actually their respective accessibility relations have this property.

the help of predicate quantifier elimination techniques as they are known from the modal logic correspondence theory. The same techniques can then be used to extract a determining property for L, the logic we are actually interested in. This all does not necessarily lead to stronger determining properties but it does so for many interesting examples. It also makes some first-order representations apparent for axiomatizations (logics) where this is not at all immediate.

The paper is now organized as follows: First, all the explanations from above are put onto formal grounds, i.e., some basic notions are introduced and the most important techniques used throughout the paper are presented. This includes a method to eliminate predicate quantifiers and a means to fix appropriate normal conservative extensions of a given logic. After that the reader is made acquainted with the main technique to find suitable determining properties (characterizations). This is followed by some application examples that show the strength but also show some weaknesses of the approach. Therefore some possible generalizations are briefly examined, although this actually belongs to future work. The paper concludes with a short summary and an outline of future extensions.

Note that the contents of this paper is not fully self-explanatory. The reader is assumed to have some knowledge on modal logics as well as on first-order predicate logic (see, e.g., [1, 3] for an introduction to modal logics).

2 Preliminaries

In this paper we only consider propositional modal logics, i.e., the (modal) formulas we are dealing with are built from propositional variables, classical connectives as, for instance, $\rightarrow$ and $\neg$ and modal operators $\Box_i$ and $\Diamond_i$ in the usual way.
By an axiomatization we understand a finite set of axioms and inference rules of the form $\vdash \Psi$ and $\vdash \Phi_1 \ldots \vdash \Phi_n \Rightarrow\ \vdash \Psi$ respectively, where Φ_j and Ψ are modal formulas. At least the classical propositional calculus (with Modus Ponens) is required for our purposes. We call such a logic[2] *normal* if for each $\Box$-operator we know that all instances of $K(\Box)$ belong to the logic, where[3]

$$K(\Box) = \left\{ \begin{array}{c} \vdash \Box(\Phi \rightarrow \Psi) \rightarrow (\Box\Phi \rightarrow \Box\Psi) \\ \vdash \Phi \ \Rightarrow\ \vdash \Box\Phi \end{array} \right\}$$

The simplest normal modal logic K is then given by an axiomatization of the classical propositional calculus (with Modus Ponens) together with $K(\Box)$. For simplicity we will use the terminology

$$K(\Box_1) + \ldots + K(\Box_n) + \left\{ \begin{array}{l} \mathrm{Ax}_1 \\ \cdots \\ \mathrm{Ax}_m \end{array} \right\}$$

to indicate that the logic under consideration knows about the modal operators $\Box_1, \ldots, \Box_n$ (and their duals $\Diamond_1, \ldots \Diamond_n$ of course), that each $\Box_i$ is a *normal* operator

[2]In the sequel we will often use the term *logic* or *logical system* when we mean an axiomatization of the above kind.

[3]Observe the difference between $\rightarrow$ and $\Rightarrow$. Whereas the former denotes logical implication the latter is used as a meta-entailment.

in the above sense and that the additional axioms $\mathrm{Ax}_1, \ldots, \mathrm{Ax}_m$ hold (the presence of a classical propositional calculus is always implicitly assumed). So, for instance, $S4$ is then given by

$$S4 = K(\Box) + \left\{ \begin{array}{l} \Box\Phi \rightarrow \Phi \\ \Box\Phi \rightarrow \Box\Box\Phi \end{array} \right\}$$

As usual, by a modal frame $\mathcal{F}$ we understand a tuple $(\mathcal{W}, \Re_i)$ where $\mathcal{W}$ denotes a non-empty set (of worlds) and each $\Re_i$ is a binary relation on $\mathcal{W}$, a so-called *accessibility relation*. A modal interpretation (or model) $\mathcal{M}$ is a frame augmented by an *actual world* τ and some valuation V which maps propositional variables to sets of worlds. Such interpretations immediately induce a *satisfiability relation* $\models$ in the usual way, e.g., a formula $\Box_i\Phi$ is true in the frame $(\mathcal{W}, \Re_i)$ under the valuation V in world τ if and only if the formula Φ holds in $(\mathcal{W}, \Re_i)$ under V in all worlds ξ such that $\Re_i(\tau, \xi)$. We write $\mathcal{F} \models \Phi$ whenever we want to indicate that $(\mathcal{F}, V, \tau) \models \Phi$ for all valuations V and worlds $\tau \in \mathcal{W}$. Finally, $\models \Phi$ holds iff $\mathcal{F} \models \Phi$ for all frames $\mathcal{F}$.

Saul Kripke [4] examined interrelations between classes of frames and logics. For instance he showed that a formula is a theorem of the logic T if and only if it is valid in all reflexive frames. We therefore say that T is *determined* by the class of all reflexive frames, or simply: T is determined by reflexivity. Quite a lot of such results have been found since then. They are particularly useful in translation-based modal theorem proving. What turned out to be especially interesting was that the resulting properties found by Kripke can sometimes be strengthened in the sense that it is often possible to find a stronger property which determines the logic just as well. A typical example can be found in the modal logic $S5$ which was shown to be determined by all frames with an equivalence as accessibility relation. Krister Segerberg showed in [10] that modal logics cannot distinguish between frames and their *generated subframes*. Providing with the whole generated subframe theory is out of the scope of this paper. Intuitively it states that we may assume that there is an initial world o such that all worlds (including o) are accessible from o by the reflexive and transitive closure of the accessibility relation (or the union of the accessibility relations if there are more than one) without violating validity or (un-)satisfiability. This generated subframe property is in itself not a first-order property since being a transitive closure of some relation cannot be represented by means of classical predicate logic. However, in case of $S5$ we know that $\Re$ is already both reflexive and transitive and thus the reflexive and transitive closure of $\Re$ is $\Re$ itself and so we may add $\exists u\ \forall w\ \Re(u, w)$ to the determining property. This – together with the equivalence relation properties for $\Re$ – then results in $\Re$ being the universal relation on the set of worlds. Thus, a formula is an $S5$ theorem iff it is valid in all equivalence frames iff it is valid in all universal frames. This information is of high importance for we can definitely work much easier with the universal relation than with equivalence relations. But there are even more such possibilities. For instance it can be shown that irreflexivity or asymmetry and many more such *negative* properties (or any weaker such properties) are not modally axiomatizable. How this can be proved shall not concern us here.[4] It implies, however,

[4]Most common is a model-theoretic proof by showing that irreflexivity is not preserved under so-called *p-morphisms* (see, e.g., [12]). Another method which is based on the unraveling technique for models can be found in [9]. Also quite interesting is a proof-theoretical approach described in [6] where it is shown that a negative property like irreflexivity could never contribute to a proof provided it is at all consistent with the rest of the occurring properties.

that modal logics cannot tell arbitrary frames from irreflexive frames and hence the modal logic K is determined by the class of irreflexive, asymmetric, etc frames.

This all might yet sound a bit vague. It therefore makes sense at this stage to provide with some useful definitions. They are given in the lines of [9].

NOTATION 1

By L_0 we understand the first-order language (with equality) on the binary predicate letters R_i (denoting accessibility relations). L_1 is L_0 with – in addition – unary predicate symbols and L_2 is L_1 with predicate quantifiers. As usual we may regard frames as ordinary relational structures for L_2. Given a L_2-formula α and a frame $\mathcal{F}$ we write $\mathcal{F} \models \alpha$ to indicate that the accessibility relations in $\mathcal{F}$ have property α, or, in other words, that $\mathcal{F}$ is an α-frame. Notice that we use the same symbol for both modal logic satisfiability and L_2-satisfiability. It will always be clear from the context which of the two is meant.

DEFINITION 1 (RELATIONAL (STANDARD) TRANSLATION $\lceil\ \rceil$)

The relational translation from modal formulas into L_1-formulas is defined by:

$$\begin{array}{lcl} \lceil P \rceil^x & = & P(x) \\ \lceil \neg\Phi \rceil^x & = & \neg\lceil \Phi \rceil^x \\ \lceil \Phi \wedge \Psi \rceil^x & = & \lceil \Phi \rceil^x \wedge \lceil \Psi \rceil^x \\ \lceil \Box_i \Phi \rceil^x & = & \forall y\ R_i(x,y) \rightarrow \lceil \Phi \rceil^y \\ \lceil \Diamond_i \Phi \rceil^x & = & \exists y\ R_i(x,y) \wedge \lceil \Phi \rceil^y \end{array}$$

with the usual homomorphic extensions for the other connectives.
Then $\lceil\Phi\rceil =_{\text{def}} \forall x \lceil\Phi\rceil^x$ and $\forall\lceil\Phi\rceil =_{\text{def}} \forall P_1, \ldots P_n\ \lceil\Phi\rceil$ where $P_1, \ldots P_n$ are all the unary predicate symbols that occur in $\lceil\Phi\rceil$.

It is well known that this relational translation behaves well, i.e., given that ϕ_L characterizes the normal modal logic L, $\models_L \Phi$ iff $\models \phi_L \rightarrow \lceil\Phi\rceil$, or, in other words, the relational translation is sound and complete.

All the above is quite standard, so let us now come to the Definitions and Lemmas that are essential for the rest of this paper. According to [9] we define:

DEFINITION 2 (SAHLQVIST)

Given a modal formula A and a frame property α we call A α-valid *whenever it holds in every frame with property α. Formally: A is called α-valid if and only if for every frame $\mathcal{F}$*

$$\mathcal{F} \models \alpha \Rightarrow \mathcal{F} \models A$$

We say that A reflects *α if the converse holds, i.e., if for every frame $\mathcal{F}$*

$$\mathcal{F} \models A \Rightarrow \mathcal{F} \models \alpha$$

DEFINITION 3 (SAHLQVIST)

A modal formula A corresponds *to an L_2-formula α if and only if*

$$\forall\mathcal{F}\ \mathcal{F} \models \alpha \Leftrightarrow \mathcal{F} \models A$$

Thus correspondence *is just the combination of validity and reflection.*
A modal logic L is determined *by α if and only if for every formula A*

$$\vdash_L A \quad \Leftrightarrow \quad (\forall \mathcal{F}\ \mathcal{F} \models \alpha \Rightarrow \mathcal{F} \models A)$$

which means that determination *is tantamount to the equivalence of provability and validity.*

Thus A trivially corresponds to $\forall\lceil A\rceil$. Moreover we get

LEMMA 1
A modal formula A is α-valid if and only if $\models \alpha \rightarrow \forall\lceil A\rceil$.
Proof: This follows immediately from the soundness and completeness of the relational translation.

$$\begin{array}{ll} \models \alpha \rightarrow \forall\lceil A\rceil & \\ \quad \text{iff} & \forall \mathcal{F}\ \mathcal{F} \models \alpha \Rightarrow \mathcal{F} \models \forall\lceil A\rceil \\ \quad \text{iff} & \forall \mathcal{F}\ \mathcal{F} \models \alpha \Rightarrow \mathcal{F} \models A \\ \quad \text{iff} & A \text{ is } \alpha\text{-valid} \end{array}$$

□

COROLLARY 1
A modal formula A corresponds *to an L_2-formula α if and only if*

$$\models \alpha \leftrightarrow \forall\lceil A\rceil$$

A modal logic L is determined *by α if and only if for every formula A*

$$\vdash_L A \quad \Leftrightarrow \quad \models \alpha \rightarrow \forall\lceil A\rceil$$

Thus *correspondence* tells us about the relation between modal axioms and frame properties whereas *determination* is concerned with the relation between modal logics and such properties. Both are particularly useful in case the property α belongs to L_0 (it is first-order). Notice that correspondence properties are always unique (up to logical equivalence) whereas determining properties need not. Whenever we have a first-order property α determining a logic L we can use any theorem prover for classical predicate logic to show the derivability of arbitrary L-theorems A and that simply by showing the first-order validity of $\alpha \rightarrow \lceil A\rceil$. It is thus of quite some importance not only to find such a first-order property (if it at all exists) but also to find one which is rather strong. There are several approaches which lead us towards such strong frame properties, witness the generated model assumption mentioned earlier which allows us to consider the universal relation instead of an equivalence relation for $S5$. As another example consider the modal logic $S4.2$ (see Section 3.3.1 on page 12) which is determined by reflexivity, transitivity, and *directedness*, where the latter is $\forall x, y, z\ R(x,y) \wedge R(x,z) \rightarrow \exists u\ R(y,u) \wedge R(z,u)$, or, in words, any two worlds with a common predecessor also have a common successor. This property can be further strengthened with the help of the generated model assumption. Notice that for $S4.2$ R is reflexive as well as transitive. We may therefore assume that $\exists u \forall v\ R(u,v)$, i.e., every world is accessible from some initial world by (the reflexive and transitive closure of) R. This, however, means that any two worlds have a common predecessor, namely this very initial world and thus the directedness property can be strengthened to *strong directedness* $\forall x, y\, \exists z\, R(x,z) \wedge R(y,z)$. Evidently, this stronger property is to be preferred over the weaker one. But can we find even stronger properties which determine $S4.2$? There are some positive answers (see, e.g., [10]) which are based on model-theoretic considerations (filtrations). The results we get from the approach proposed in this paper are similar, the way how they are obtained is different, though.

Definition 4

For any logic L let $\mathcal{L}_L$ be the set of formulas (the language) of L. We call a logic L^{ex} a normal conservative extension *of L if L^{ex} is a normal modal logic with $\mathcal{L}_L \subseteq \mathcal{L}_{L^{\mathrm{ex}}}$ and for every $A \in \mathcal{L}_L$: $\vdash_L A$ iff $\vdash_{L^{\mathrm{ex}}} A$, i.e., all L^{ex}-theorems that belong to $\mathcal{L}_L$ are L-theorems as well.*

Evidently, if N is a normal conservative extension of M and M is a normal conservative extension of L then N is a normal conservative extension of L.

A logic is sound with respect to α whenever its axiomatization is α-valid. On the other hand, completeness with respect to α can be shown for a logic L if L's axioms reflect α and L is *adequate*, which means that for every non-theorem F of L there exists a frame that validates each derivable (in L) formula but that does not validate F. Hence correspondence implies determination if the logic under consideration is adequate. Since corresponding properties are unique up to equivalence this is of little use for us for we actually want to strengthen determining properties and stronger determining properties can impossibly correspond to the axiomatization anymore. We therefore emphasize on the correspondence property for the extended logic's axiomatization and from that try to extract a determining property for the original logic.

Definition 5

Let α be an L_2-formula and let L be a normal modal logic with operators $\Box_1, \ldots, \Box_n$. We denote the accessibility relation symbols associated with $\Box_i$ in the Kripke-style semantics for L by R_i. Now let $\{S_1, \ldots, S_m\}$ be the set of all binary accessibility relation symbols that occur in α but not in $\{R_1, \ldots, R_n\}$. Then by $\exists^L \alpha$ we understand $\exists S_1, \ldots, S_m\, \alpha$, which means that $\exists^L \alpha$ is α with an existential quantification for each "unknown" (to L) binary relation symbol.

As a little example consider a logic L which knows about the operators $\Box$ and $\blacksquare$ and let α be an L_2-formula over the binary relation symbols R, S, and T where R and S are the accessibility relation symbols associated with $\Box$ and $\blacksquare$ respectively. Then $\exists^L \alpha = \exists T\, \alpha$ since T is the only accessibility relation symbol that does not belong to some modal operator in L.

Lemma 2

For any $A \in \mathcal{L}_L$:

$$\models \alpha \to \forall \lceil A \rceil \quad \textit{iff} \quad \models \exists^L \alpha \to \forall \lceil A \rceil$$

Proof: Let $S_1, \ldots, S_m$ be the binary relation symbols occurring in α but not in $\forall \lceil A \rceil$

and let $R_1, \ldots, R_n$ be the binary relation symbols occurring in $\forall\lceil A \rceil$. Then

$$\begin{array}{ll} & \models \alpha \rightarrow \forall\lceil A \rceil \\ \text{iff} & \models \forall R_1, \ldots, R_n, S_1, \ldots, S_m\ (\alpha \rightarrow \forall\lceil A \rceil) \\ \text{iff} & \models \forall R_1, \ldots, R_n\ ((\exists S_1, \ldots, S_m\ \alpha) \rightarrow \forall\lceil A \rceil) \\ & \quad \text{since } S_1, \ldots, S_m \text{ do not occur in } \forall\lceil A \rceil \\ \text{iff} & \models \exists^L \alpha \rightarrow \forall\lceil A \rceil \\ & \quad \text{by definition of } \exists^L \end{array}$$

□

THEOREM 1
If L^{ex} is a normal conservative extension of L and L^{ex} is determined by α then L is determined by $\exists^L \alpha$.
Proof: L^{ex} is determined by α, thus for all $A \in \mathcal{L}_{L^{\text{ex}}}$:

$$\vdash_{L^{\text{ex}}} A \quad \Leftrightarrow \quad \models \alpha \rightarrow \forall\lceil A \rceil$$

Now consider an arbitrary $B \in \mathcal{L}_L$. Then $B \in \mathcal{L}_{L^{\text{ex}}}$ and

1. $\vdash_L B$ iff $\vdash_{L^{\text{ex}}} B$ (L^{ex} is a normal conservative extension of L)
2. $\models \alpha \rightarrow \forall\lceil B \rceil$ iff $\models \exists^L \alpha \rightarrow \forall\lceil B \rceil$ (by Lemma 2)

Hence

$$\vdash_L B \quad \Leftrightarrow \quad \vdash_{L^{\text{ex}}} B \quad \Leftrightarrow \quad \models \alpha \rightarrow \forall\lceil B \rceil \quad \Leftrightarrow \quad \models \exists^L \alpha \rightarrow \forall\lceil B \rceil$$

and so L is determined by $\exists^L \alpha$. □

3 The Approach

Theorem 1 tells us what has to be achieved. Given a logic L we try to find a normal conservative extension L^{ex} of L, search for a determining property α for L^{ex} and then compute $\exists^L \alpha$. Remains the question how to find suitable normal conservative extensions. And even if the search for this was successful, how can we possibly fix α and $\exists^L \alpha$? For the latter problem there are some well-known solutions nowadays. Recall that it suffices to find correspondence properties for a given axiomatization provided the logic under consideration is adequate. Adequacy, on the other hand, can sometimes even be shown syntactically, witness the so-called *Sahlqvist formulas* as they are described in [9]. Hence we can apply known *second-order quantifier elimination techniques* for solving both problems, finding α and extracting $\exists^L \alpha$. There are various such elimination methods available today. Best known are probably the approaches by Sahlqvist [9], van Benthem [12], and Gabbay and Ohlbach [2].[5] More recently a further elimination method has been published in [7] which at least subsumes the former two. It is based on a fixpoint approach as described below.

[5] An implementation (called SCAN) is available on the world-wide-web at http://www.mpi-sb.mpg.de/guide/staff/ohlbach/scan/scan.html/.

3.1 Second-Order Quantifier Elimination

We say that a formula Φ is positive w.r.t. some predicate symbol P if the negation normal form[6] of Φ contains only positive P-literals. By $\Phi(P)$ we indicate that P is a free predicate letter in Φ. This notation allows us to write $\Phi(Q)$ in order to specify Φ with every occurrence of P replaced by Q. In particular, $\Phi(\top)$ then means Φ with every occurrence of P replaced by `true`. We call this a *predicate substitution.* Occasionally we also need two other kinds of substitutions, a *sub-formula substitution* and the usual *variable substitution.* The former is represented by $\Phi[P(\alpha) \leftarrow Q(\beta)]$ where this means that we are considering Φ with every occurrence of the sub-formula $P(\alpha)$ replaced by $Q(\beta)$. The notation for variable substitutions is as usual, i.e., we write Φ^x_β whenever we want to express that we are considering Φ with every occurrence of the variable x replaced by the term β.

THEOREM 2 (ELIMINATION THEOREM)
Let $\Phi(P)$ and Ψ both be positive with respect to P. Then

$$\exists P \ \ (\forall x\ (P(x) \rightarrow \Phi(P)) \wedge \Psi)$$
$$\leftrightarrow$$
$$\Psi\left[P(\alpha) \leftarrow \left[\nu P(x).\, \Phi(P)\right]^x_\alpha\right]$$

where $\nu P(x).\Phi(P)$ is a fixpoint formula representing the infinite conjunction

$$\nu P(x).\Phi(P) = \bigwedge_{i \in \omega} \Phi^i(\top)$$

with $\Phi^{i+1}(\top) = \Phi(\Phi^i(\top))$ and $\Phi^0(\top) = \top$.
Proof: can be found in [7]. □

Evidently, the above theorem also holds if P is replaced by $\neg P$ and Φ and Ψ both are negative w.r.t. P.
Actually, for the purpose of this paper it is more general than necessary. Note that in case Φ has no mention of P at all then $\nu P(x).\, \Phi(P)$ is just Φ itself and we end up in the special case described by Andrzej Szałas in [11].[7]

We want to make use of the Elimination Theorem in order to find first-order equivalents for formulas $\forall\lceil A \rceil$ and $\exists^L \forall \lceil A \rceil$ respectively. Such formulas are obviously not always of the form required for the application of the Elimination Theorem. However, it can quite easily be shown (see [7]) that we are always able to produce the appropriate form provided we allow (for some cases) second-order skolemization. In the worst case this results in another (or even identical) second-order formula. This should not surprise us too much for we certainly cannot transform every L_2-formula into an equivalent formula of (infinite) L_0.

If we succeeded in such a suitable transformation without generating Skolem functions[8] we obtain a fixpoint formula which can be "computed", which means that either

[6] The negation normal form is obtained after elimination of all implication and equivalence signs and moving the negation symbols inwards as far as possible.

[7] This version is also accessible via the world-wide-web. See http://www.ida.liu.se/labs/kplab/projects/dls/ for the implementation and further literature.

[8] There are cases, however, were Skolem functions are to be introduced but nevertheless a final "deskolemization" is possible.

this fixpoint formula is bounded and therefore can be transformed into an equivalent first-order formula or we end up with an infinite formula.[9]

Further details about this elimination technique shall not concern us here. The interested reader is again referred to [7].

3.2 Finding Normal Conservative Extensions

Two major problems have to be solved for the approach presented in this paper: First, we have to be able to eliminate predicate quantifiers and second, a means is needed which helps us to find suitable normal conservative extensions of a given logic. The former problem can be solved with the help of the Elimination Theorem. Remains the question how we can find such a suitable extension. The idea we are following here has to do with the definition of auxiliary modal operators which are based on existing operators. Such a definition will usually have some influence on what becomes provable and what not. What is to be guaranteed is that the "newly provable theorems" do not belong to the language of the original logic. This can be achieved by a suitable choice of *modalities*.

Definition 6 (Modalities)
Given a logical system L which knows about the modal operators $\Box_1, \ldots, \Box_n$ and $\Diamond_1, \ldots, \Diamond_n$ we understand by a modality $\mathbf{M}$ *any finite sequence with members taken from $\{\Box_i \mid 1 \leq i \leq n\} \cup \{\Diamond_i \mid 1 \leq i \leq n\}$.*
Such a modality is called normal *w.r.t. L if $K(\mathbf{M})$ holds, i.e., if*

1. $\vdash_L \mathbf{M}(\Phi \rightarrow \Psi) \rightarrow (\mathbf{M}\,\Phi \rightarrow \mathbf{M}\,\Psi)$
2. $\vdash_L \Phi \Rightarrow\ \vdash_L \mathbf{M}\,\Phi$

An auxiliary modal operator $\blacksquare$ *is defined by the axiom $\blacksquare\Phi \leftrightarrow \mathbf{M}\,\Phi$, where $\mathbf{M}$ is a modality (not necessarily* normal*). Again we use $\blacklozenge$ as an abbreviation for $\neg\blacksquare\neg$.*

From a pure syntactic point of view, adding such a defining axiom won't let us prove more formulas of the original language than before. However, what we have lost is a nice Kripke-style possible world semantics for the new logic then for this new operator is not necessarily *normal*. Simply forcing the new operator to be normal is but a dangerous thing to do, though. After all, this would mean that for any provable formula Φ the formula $\mathbf{M}\,\Phi$ becomes provable as well even though this may not have been the case without this new operator. We therefore have to be aware of the "extra formulas" that get provable, or, to turn it the other way round, we ought to choose the modality accordingly such that the corresponding K-axiom and necessitation rule cannot lead to anything new what the original logic language is concerned. This then means that $K(\mathbf{M})$ is to be derivable already in the given logic. It may not at all be immediate why this can lead to any interesting results. Nevertheless it does and why this is so is shown below.

[9] Witness the famous Löb-Axiom $\Box(\Box\Phi \rightarrow \Phi) \rightarrow \Box\Phi$ which results in transitivity together with (the non-first-order) backward well-foundedness.

LEMMA 3
Given a normal modal logic L and a modality $\mathbf{M}$ which is normal w.r.t. L. Then

$$L^{\text{ex}} = L + K(\blacksquare) + \{\blacksquare\Phi \leftrightarrow \mathbf{M}\,\Phi\}$$

is a normal conservative extension of L.
Proof: L^{ex} is obviously a normal extension of L since it contains both L and $K(\blacksquare)$. Remains to be shown that for every $A \in \mathcal{L}_L : \vdash_{L^{\text{ex}}} A \Rightarrow \vdash_L A$. To this end we define a translation $\Pi\colon \mathcal{L}_{L^{\text{ex}}} \to \mathcal{L}_L$ as

$$\begin{array}{lcl} \Pi(P) & = & P, \text{ for propositional letters } P \\ \Pi(\blacksquare\Phi) & = & \mathbf{M}\,\Pi(\Phi) \\ \Pi(\blacklozenge\Phi) & = & \neg\,\mathbf{M}\,\neg\Pi(\Phi) \end{array}$$

where the other operators and connectives are treated by the homomorphic extension of the above.
Now, for every $A \in \mathcal{L}_{L^{\text{ex}}}$, a simple induction on the derivation of A shows that[10]

$$\vdash_{L^{\text{ex}}} A \Rightarrow\ \vdash_L \Pi(A)$$

Note that $\Pi(B) = B$, for every $B \in \mathcal{L}_L$. Therefore

$$\vdash_{L^{\text{ex}}} B \Rightarrow\ \vdash_L \Pi(B) \Leftrightarrow\ \vdash_L B$$

and we are done. □

At this stage we have a means at hand to fix appropriate normal conservative extensions L^{ex} of a given normal modal logic L. Also there are possibilities to extract correspondence and determination properties for L^{ex}. Thus, according to Lemma 3, we have everything that is needed to find some determination properties for L. This will be exemplified in the sections to follow.

3.3 Application Examples

It is sometimes not very easy to check whether the auxiliary modality or operator is normal or not. In fact, we do not know of many general results on this. The following one turns out to be useful, though.

LEMMA 4
Let $\mathbf{M}$ be a modality made up of $\Box$ and $\Diamond$ in a modal logic extending KD4, i.e., the axioms $\Box\Phi \to \Box\Box\Phi$ and $\Box\Phi \to \Diamond\Phi$ are contained or at least derivable in the logic under consideration.
If $\vdash \mathbf{M}\,\Phi \to \neg\,\mathbf{M}\,\neg\Phi$ *then* $\mathbf{M}$ *is a normal modality.*
Proof: We have to show that $K(\mathbf{M})$ holds.
Since $\Box$ is a normal modality with $\vdash \Phi \Rightarrow \vdash \Diamond\Phi$ (this holds in KD already), a trivial induction on the length of $\mathbf{M}$ leaves $\vdash \Phi \Rightarrow \vdash \mathbf{M}\,\Phi$.
For the K-axiom (for $\mathbf{M}$) it suffices to show that

$$\mathbf{M}\,A \wedge \mathbf{M}\,B \to \mathbf{M}(A \wedge B)$$

[10] For a detailled proof see [5].

for if B is set to $A \to C$ we obtain the desired result.
The case were $\mathbf{M}$ is of the form $\Box\Box\cdots\Box$ is trivial. Hence assume that $\mathbf{M}$ is of the form $\Diamond^{n_1}\Box^{m_1}\cdots\Diamond^{n_k}\Box^{m_k}$ with at least one $\Diamond$. Then – with the help of the axiom $\Box\Phi \to \Box\Box\Phi$ – it follows that

$$\mathbf{M}\,A \to \Diamond^{n_1}\Box^{m_1}\Box^{m_1}\Diamond^{n_2}\Box^{m_2}\Box^{m_2}\cdots\Diamond^{n_k}\Box^{m_k}\Box^{m_k}\ A$$

Moreover we are able to derive[11]

$$\mathbf{M}\,B \to \Box^{n_1}\Diamond^{m_1}\Box^{m_1}\Box^{n_2}\Diamond^{m_2}\Box^{m_2}\Box^{n_3}\cdots\Box^{m_{k-1}}\Box^{n_k}\Diamond^{m_k}\Box^{m_k}B$$

Now recall that for every normal modal logic we know that $\Box\Phi \wedge \Diamond\Psi \to \Diamond(\Phi \wedge \Psi)$ and $\Box\Phi \wedge \Box\Psi \to \Box(\Phi \wedge \Psi)$. Hence

$$\mathbf{M}\,A \wedge \mathbf{M}\,B \to \Diamond^{n_1}\Diamond^{m_1}\Box^{m_1}\Diamond^{n_2}\Diamond^{m_2}\Box^{m_2}\cdots\Diamond^{n_k}\Diamond^{m_k}\Box^{m_k}\ (A \wedge B)$$

which can be simplified by $\Diamond\Diamond\Phi \to \Diamond\Phi$ to

$$\mathbf{M}\,A \wedge \mathbf{M}\,B \to \Diamond^{n_1}\Box^{m_1}\Diamond^{n_2}\Box^{m_2}\cdots\Diamond^{n_k}\Box^{m_k}\ (A \wedge B)$$

and thus to $\mathbf{M}\,A \wedge \mathbf{M}\,B \to \mathbf{M}(A \wedge B)$. $\Box$

3.3.1 Logics extending $S4.2$

Let us have a look at the modal logic $S4.2$ which extends $S4$ with the additional axiom $\Diamond\Box\Phi \to \Box\Diamond\Phi$. As it is known[12] from the literature this axiom corresponds to the so-called *directedness* or *Diamond–Property* which states that any two worlds with a common predecessor also have a common successor. It is thus a question of confluence that is described here. Formally, we consider the following axiomatization

$$L = S4.2 = K(\Box) + \left\{ \begin{array}{l} \Box\Phi \to \Phi \\ \Box\Phi \to \Box\Box\Phi \\ \Diamond\Box\Phi \to \Box\Diamond\Phi \end{array} \right\}$$

and its corresponding theory (which also determines $S4.2$)

$$\begin{array}{l} \forall u\ R(u,u) \\ \forall u,v,w\ R(u,v) \wedge R(v,w) \to R(u,w) \\ \forall u,v,w\ R(u,v) \wedge R(u,w) \to \exists x\ R(v,x) \wedge R(w,x) \end{array}$$

Our aim is now to strengthen this theory, i.e., we are interested in a more general property which also determines $S4.2$. To this end consider the following logical system

$$L^{\text{ex}} = K(\Box) + K(\blacksquare) + \left\{ \begin{array}{l} \Box\Phi \to \Phi \\ \Box\Phi \to \Box\Box\Phi \\ \Diamond\Box\Phi \to \Box\Diamond\Phi \\ \blacksquare\Phi \leftrightarrow \Diamond\Box\Phi \end{array} \right\}$$

[11] The proof of this fact is easy but tedious. It requires $\Box\Phi \to \Box\Box\Phi$ as well as $\mathbf{M}\,\Phi \to \neg\mathbf{M}\neg\,\Phi$. The interested reader is again referred to [5].
[12] We might of course find out about this with the help of the Elimination Theorem.

$S4.2$ extends $KD4$ and, according to Lemma 4 and the presence of the axiom $\Diamond\Box\Phi \rightarrow \Box\Diamond\Phi$, $\Diamond\Box$ is a normal modality w.r.t. $S4.2$. Hence, by Lemma 3, L^{ex} is a normal conservative extension of L.
Now let us have a look at the determining property for L^{ex}. What the first three axioms are concerned, no difficulties do arise; they just correspond to reflexivity, transitivity, and the above mentioned "Diamond-Property". For the fourth axiom we have to invoke the Elimination Theorem, i.e., we have to compute

$$\forall \Phi, x \ \ [(\forall y\, S(x,y) \rightarrow \Phi(y)) \leftrightarrow (\exists u\, R(x,u) \wedge \forall v\, R(u,v) \rightarrow \Phi(v))] \tag{1}$$

This finally results in the conjunction of two formulas

$$\forall x\, \exists y\, (R(x,y) \wedge (\forall z\, (R(y,z) \rightarrow S(x,z)))) \tag{2}$$

$$\forall x\, \forall y\, (R(x,y) \rightarrow \forall z\, (S(x,z) \rightarrow R(y,z))) \tag{3}$$

These two formulas denote the determining property (the α) of L^{ex}. Thus, according to Theorem 1, it suffices to find a first-order equivalent for $\exists S\, \alpha$. Again this is a case for the Elimination Theorem which yields

$$\forall x\, \exists e\, (R(x,e) \wedge \forall y\, (R(x,y) \rightarrow \forall z\, (R(e,z) \rightarrow R(y,z)))) \tag{4}$$

This result seems somewhat awkward at the first glance. However, under seriality and transitivity of R and the generated model assumption for normal modal logics it is equivalent to

$$\exists e\, \forall x\, R(x,e) \tag{5}$$

as can be checked with any standard first-order logic theorem prover. This property is usually called *finality* and we end up with

THEOREM 3
The $S4.2$ frames are determined by reflexivity, transitivity, and finality.

Actually, reflexivity was not really needed in order to obtain finality; seriality alone would have sufficed.[13] Thus we get as a corollary:

COROLLARY 2
Finality may consistently be assumed for any extension of $KD4.2$.

Krister Segerberg defines in [10] a *non-degenerated cluster* as a collection of worlds such that for any two worlds u and v taken from this cluster u is accessible from v and vice versa. He then claims that one can prove with the help of so-called *Lemmon-filtrations* that $KD4.2$ is determined by seriality, transitivity, and the existence of a *last non-degenerated cluster*. Now, if we take a closer look at our finality-property we realize that finality expresses just the fact that there is a final non-degenerated cluster. Thus the approach proposed here mirrors to some extent what has otherwise been achieved with the help of certain special filtration techniques.

[13] Note that in this case the generated model assumption is to be described by $\exists u\ \forall v\ v = u \vee R(u,v)$, i.e., the reflexive closure of R.

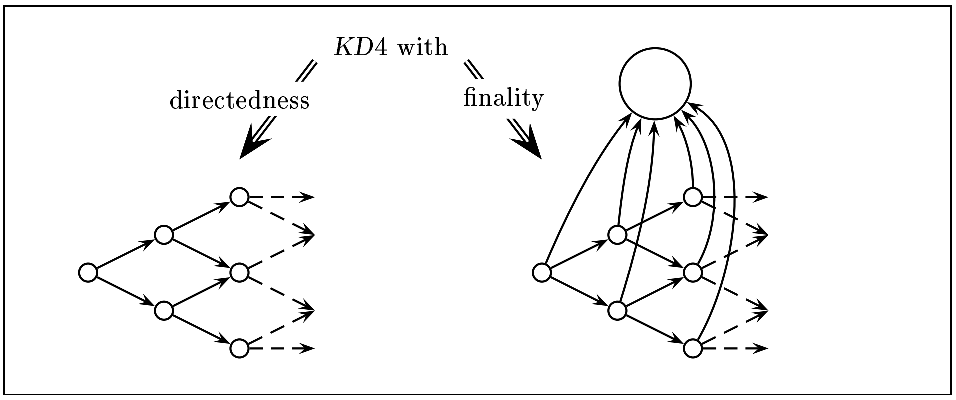
Observe the difference between $KD4.2$ structures and $KD4$ structures with a final cluster as given in the following figure.[14] It reflects the difference of the two background theories

$$\begin{gathered} \forall u \; \exists v \; R(u,v) \\ \forall u,v,w \; R(u,v) \wedge R(v,w) \rightarrow R(u,w) \\ \forall u,v,w \; R(u,v) \wedge R(u,w) \rightarrow \exists x \; R(v,x) \wedge R(w,x) \end{gathered}$$

and

$$\begin{gathered} \exists u \; \forall w \; R(w,u) \\ \forall u,v,w \; R(u,v) \wedge R(v,w) \rightarrow R(u,w) \end{gathered}$$

which are indistinguishable for modal logics (they both characterize $KD4.2$).



The above normal conservative extension is evidently not the only one we can imagine. For some reason which will become clear in the next section it is even recommended that other normal conservative extensions should be searched for in general. As an example let us have a look at the following alternative logical system for $KD4.2$.

$$L^{\mathrm{ex}} = K(\Box) + K(\blacksquare) + \left\{ \begin{array}{l} \Box\Phi \rightarrow \Phi \\ \Box\Phi \rightarrow \Box\Box\Phi \\ \Diamond\Box\Phi \rightarrow \blacksquare\Phi \\ \blacksquare\Phi \rightarrow \blacklozenge\Phi \end{array} \right\}$$

This again is a normal conservative extension of L since, under $\Diamond\Box\Phi \leftrightarrow \blacksquare\Phi$, the schemata $\Diamond\Box\Phi \rightarrow \Box\Diamond\Phi$ and $\blacksquare\Phi \rightarrow \blacklozenge\Phi$ are logically equivalent and so the latter can replace the former. Moreover, the one direction of the equivalence $\Diamond\Box\Phi \leftrightarrow \blacksquare\Phi$ is not needed in order to remain a normal conservative extension as can easlity be checked.

Hence, what has to be done next is to extract a determining property α for this new L^{ex}. To this end it suffices to find their respective correspondence properties.[15] For the first two schemata there are again no problems; they correspond to reflexivity and transitivity of the accessibility relation R. Most readers probably are familiar

[14] Transitivity edges are omitted for readability. Smaller circles denote worlds and the bigger circle represents a cluster.

[15] All schemata we are dealing with are so-called *Sahlqvist-formulas* and therefore the conjunction of the respective correspondence properties may serve as a determining property.

with the fourth schema as well: It corresponds to seriality of S. And what the third schema is concerned: From the above we know already the property it corresponds to, namely $\forall u, v, w\ R(u,v) \wedge S(u,w) \to R(v,w)$. We thus have found the α we were looking for. Remains to determine $\exists^L \alpha$.

LEMMA 5
L is determined by reflexivity, transitivity and

$$\forall u\ \exists v\ \forall w\ R(u,w) \to R(w,v)$$

Proof: According to our general procedure we have to compute

$$\exists S \left[\begin{array}{l} \forall u, v, w\ R(u,v) \wedge S(u,w) \to R(v,w) \\ \forall u \exists v\ S(u,v) \end{array} \right]$$

which the Elimination Theorem shows to be equivalent to what has been claimed. □

Recall that we may consider the generated model assumption and in doing so – by adding the formula $\exists u\ \forall v\ R(u,v)$ – this property can be further simplified to

$$\exists x\ \forall y\ R(y,x)$$

i.e., *finality*. Thus we ended up at the same result as we did with the former normal conservative extension. Why the latter kind of extension is often nevertheless to be preferred is going to be exemplified in the following section.

3.3.2 Logics extending $S4.1$

Let us consider modal logics which extend $S4.1$, i.e., we assume that the following axioms are contained in (or are at least derivable from) the axiomatization we are interested in.

$$L = S4.1 = K(\Box) + \left\{ \begin{array}{l} \Box\Phi \to \Phi \\ \Box\Phi \to \Box\Box\Phi \\ \Box\Diamond\Phi \to \Diamond\Box\Phi \end{array} \right\}$$

The problem with the third axiom is that it does not correspond to a first-order property. Nevertheless, under the transitivity-axiom $\Box\Phi \to \Box\Box\Phi$ this logical system *is determined* by a first-order property (see [12]). The proof of this fact is somewhat hard to grasp, though.

Now, according to our general recipe we now determine a suitable auxiliary modality. This can be found in $\mathbf{M} = \Box\Diamond$ and we define

$$L^{\mathrm{ex}} = K(\Box) + K(\blacksquare) + \left\{ \begin{array}{l} \Box\Phi \to \Phi \\ \Box\Phi \to \Box\Box\Phi \\ \Box\Diamond\Phi \to \blacksquare\Phi \\ \blacksquare\Phi \to \blacklozenge\Phi \end{array} \right\}$$

LEMMA 6
L^{ex} is a normal conservative extension of L.

Proof: L^{ex} evidently extends L because $\Box\Diamond\Phi \to \Diamond\Box\Phi$ follows almost immediately from $\blacksquare\Phi \to \blacklozenge\Phi$ and $\Box\Diamond\Phi \to \blacksquare\Phi$.

Now, notice that $S4.1$ (and thus each of its extensions) extends $KD4$. Moreover – because of the presence of the axiom $\Box\Diamond\Phi \to \Diamond\Box\Phi$ – we know that $\mathbf{M}\,\Phi \to \neg\,\mathbf{M}\neg\,\Phi$. Hence, from Lemmas 4 and 3 it follows that

$$L_{+}^{\mathrm{ex}} = L + \{\blacksquare\Phi \leftrightarrow \Box\Diamond\Phi\}$$

is a normal conservative extension of L. Now, because of this added equivalence, it is allowed to replace the axiom $\Box\Diamond\Phi \to \Diamond\Box\Phi$ with $\blacksquare\Phi \to \blacklozenge\Phi$ and we still have a normal conservative extension. Finally, note that L^{ex} is a sub-logic of L_{+}^{ex} and so for every $A \in \mathcal{L}_L$:

$$\vdash_{L^{\mathrm{ex}}} A \Rightarrow\ \vdash_{L_{+}^{\mathrm{ex}}} A \Rightarrow\ \vdash_L A \Rightarrow\ \vdash_{L^{\mathrm{ex}}} A$$

and we are done. $\Box$

After we have found a normal conservative extension L^{ex} of L we turn to the problem of finding a determining property for L^{ex}. Fortunately, all axioms in L^{ex} are so-called Sahlqvist-formulas (see [9]) such that a determining property is already given by the combination of the respective correspondence properties for the axiomatization. It thus suffices to find these correspondence properties for each axiom in L^{ex}.

LEMMA 7

Let S denote the accessibility relation associated with $\blacksquare$. The axioms of L^{ex} then correspond to reflexivity and transitivity of R and

$$\begin{array}{l} \forall v \exists w\ S(v,w) \\ \forall x,u\ S(x,u) \to \exists y\ (R(x,y) \wedge \forall z\ (R(y,z) \to z = u)) \end{array}$$

Proof: It is well known that $\Box\Phi \to \Phi$ corresponds to reflexivity of R and that $\Box\Phi \to \Box\Box\Phi$ corresponds to transitivity of R. For the remaining two axioms we have to compute $\forall\Phi, x\ \lceil \blacksquare\Phi \to \blacklozenge\Phi \rceil^x$ and $\forall\Phi, x\ \lceil \Box\Diamond\Phi \to \blacksquare\Phi \rceil^x$ respectively. With the first of the two there are no problems for it obviously results in the seriality of S, i.e., in $\forall v \exists w\ S(v,w)$. In order to solve the second problem we apply the Elimination Theorem.

$$\begin{array}{c} \forall x, \Phi\ \lceil \Box\Diamond\Phi \to \blacksquare\Phi \rceil^x \\ \leftrightarrow \\ \neg\exists x, \Phi\ \lceil \Box\Diamond\Phi \wedge \neg\blacksquare\Phi \rceil^x \\ \leftrightarrow \\ \neg\exists x, \Phi \left[\begin{array}{l} \forall y\ R(x,y) \to \exists z\ R(y,z) \wedge \Phi(z) \\ \exists u\ S(x,u) \wedge \neg\Phi(u) \end{array} \right] \\ \leftrightarrow \\ \neg\exists u, x, \Phi \left[\begin{array}{l} \forall v\ \neg\Phi(v) \vee u \neq v \\ S(x,u) \\ \forall y\ R(x,y) \to \exists z\ R(y,z) \wedge \Phi(z) \end{array} \right] \\ \leftrightarrow \\ \neg\exists u, x \left[\begin{array}{l} S(x,u) \\ \forall y\ R(x,y) \to \exists z\ R(y,z) \wedge z \neq u \end{array} \right] \\ \leftrightarrow \\ \forall x,u\ S(x,u) \to \exists y\ (R(x,y) \wedge \forall z\ (R(y,z) \to z = u)) \end{array}$$

and we are done. $\square$

The properties obtained in the previous Lemma determine L^{ex}. Their conjunction therefore serves as the α of Theorem 1. From this we can quite easily extract a determining property for L, which is $\exists^L\alpha$. Recall that, according to the definition of $\exists^L$, we have to find out which binary relations have no associated $\square$-operator in the language of L. In the case we are just considering this is the relation symbol S and so $\exists^L\alpha$ is the conjunction of reflexivity, transitivity and

$$\exists S \left[\begin{array}{c} \forall v \exists w\ S(v,w)\ \wedge \\ \forall x,u\ S(x,u) \rightarrow \exists y\ (R(x,y) \wedge \forall z\ (R(y,z) \rightarrow z = u)) \end{array} \right]$$

This is another case for the Elimination Theorem and its application results in

$$\forall v \exists w,y\ R(v,y) \wedge \forall z\ (R(y,z) \rightarrow z = w)$$

This looks fairly complicated at the first glance, however, given transitivity, it is equivalent to the slightly simpler

$$\forall u \exists v\ R(u,v) \wedge \forall w\ (R(v,w) \rightarrow v = w)$$

as can easily be proved with the help of any standard predicate logic theorem prover. This latter formula describes the so-called *atomicity* of the underlying structure and so we finally end up with

THEOREM 4
The S4.1 frames are determined by reflexivity, transitivity, and atomicity.

Notice that reflexivity was actually never needed in the above proofs and so atomicity holds already for $K4.1$ frames. Moreover, for any given logic that extends $K4.1$[16] the above procedure can be performed and thus for any such logic atomicity may be assumed as one of the frame properties that determine the logic. Hence we may conclude

COROLLARY 3
Atomicity may consistently be assumed for any logic extending K4.1.

3.3.3 Logics extending *weak dense* $KD4.3$

As a further example let us have a look at the logic *weak dense* $KD4.3$, i.e., we consider

$$L = K(\square) + \left\{ \begin{array}{c} \square\Phi \rightarrow \diamond\Phi \\ \square\Phi \rightarrow \square\square\Phi \\ \square\square\Phi \rightarrow \square\Phi \\ \diamond\Phi \wedge \diamond\Psi \rightarrow \diamond(\Phi \wedge \diamond\Psi) \vee \diamond(\Phi \wedge \Psi) \vee \diamond(\diamond\Phi \wedge \Psi) \end{array} \right\}$$

These axioms correspond to *seriality*, *transitivity*, *density*, and *right-linearity* of the accessibility relation R.[17] At the first glance $\square\square$ seems a good candidate as an

[16] which means that the axioms $\square\Phi \rightarrow \square\square\Phi$ and $\square\diamond\Phi \rightarrow \diamond\square\Phi$ are at least derivable

[17] Density means that $\forall x,y\ R(x,y) \rightarrow \exists z\ (R(x,z) \wedge R(z,y))$. In the standard literature this property is often called *weak density* for it does not really guarantee that the worlds "in-between" x and y are different from x and y respectively. For instance it holds already under $\square\Phi \rightarrow \Phi$, the reflexivity-axiom.

auxiliary modality. However, it turns out that this does not lead to anything of interest (as would any auxiliary modal operator defined by an arbitrary sequence of the form $\Box \cdots \Box$. In order to overcome this problem we first extend the logic L with the help of another modal operator as

$$L_1^{\mathrm{ex}} = L + K(\boxminus) + \left\{ \begin{array}{c} \Phi \to \Box ⟠ \Phi \\ \Phi \to \boxminus \Diamond \Phi \end{array} \right\}$$

Let $R^{\smile}$ be the accessibility relation symbol associated with the operator $\boxminus$. Then the two additional axioms correspond to the property $\forall x, y\ R(x,y) \leftrightarrow R^{\smile}(y,x)$, i.e., $R^{\smile}$ is just the converse of R.[18] Therefore, L_1^{ex} is evidently a normal conservative extension of L. Now, within L_1^{ex} the axioms $\Box\Phi \to \Box\Box\Phi$, $\Box\Box\Phi \to \Box\Phi$, and $\Diamond\Phi \wedge \Diamond\Psi \to \Diamond(\Phi \wedge \Diamond\Psi) \vee \Diamond(\Phi \wedge \Psi) \vee \Diamond(\Diamond\Phi \wedge \Psi)$ can be replaced by $⟠\Phi \to \Box⟠\Phi$, $\Diamond\boxminus\Phi \to \Diamond\Phi$, and $⟠\Diamond\Phi \to (⟠\Phi \vee \Phi \vee \Diamond\Phi)$ respectively, for they correspond to the same first-order property.

Now it turns out that $\Diamond\boxminus$ is normal w.r.t. L_1^{ex} as can easily be checked with any standard theorem prover.[19] This finally leads us to

Lemma 8
The logic

$$L^{\mathrm{ex}} = K(\Box) + K(\boxminus) + K(\blacksquare) + \left\{ \begin{array}{c} \Phi \to \Box⟠\Phi \\ \Phi \to \boxminus\Diamond\Phi \\ \blacklozenge\Phi \to \Box⟠\Phi \\ ⟠\Phi \to \blacklozenge\Phi \\ \blacksquare\Phi \to \Diamond\Phi \\ ⟠\Diamond\Phi \to (⟠\Phi \vee \Phi \vee \Diamond\Phi) \end{array} \right\}$$

is a normal conservative extension of weak dense $KD4.3$.

Proof: Since $\Diamond\boxminus$ is a normal modality, we can introduce a new modal operator $\blacksquare$ and define it by $\blacksquare\Phi \leftrightarrow \Diamond\boxminus\Phi$. With this we reformulate the axioms $⟠\Phi \to \Box⟠\Phi$ and $\Diamond\boxminus\Phi \to \Diamond\Phi$ as $⟠\Phi \to \blacklozenge\Phi$ and $\blacksquare \to \Diamond\Phi$ respectively, and so we know that

$$L_2^{\mathrm{ex}} = K(\Box) + K(\boxminus) + K(\blacksquare) + \left\{ \begin{array}{c} \Box\Phi \to \Diamond\Phi \\ \Phi \to \Box⟠\Phi \\ \Phi \to \boxminus\Diamond\Phi \\ \blacksquare\Phi \leftrightarrow \Diamond\boxminus\Phi \\ ⟠\Phi \to \blacklozenge\Phi \\ \blacksquare\Phi \to \Diamond\Phi \\ ⟠\Diamond\Phi \to (⟠\Phi \vee \Phi \vee \Diamond\Phi) \end{array} \right\}$$

is a normal conservative extension of *weak dense* $KD4.3$.

Now, $\Box\Phi \to \Diamond\Phi$ already follows from $\blacksquare\Phi \to \Diamond\Phi$, $K(\blacksquare)$, and Modus Ponens and is therefore redundant. Moreover, it turns out that one part of the defining equivalence,

[18] The reader familiar with temporal or tense logics might recognize $\boxminus$ and $⟠$ as something like past-operators.

[19] Observe that this modality is normal *because* of the presence of the right-linearity axiom. Without this extra property this modality would fail to be normal.

$\blacksquare\Phi \to \Diamond\boxminus\Phi$, is not necessary to ensure that the remaining logical system is a normal conservative extension of L, for every axiom of L can be derived even without it. This follows easily from

$$\begin{gathered} \diamondminus\Phi \to \blacklozenge\Phi \to \Box\diamondminus\Phi \\ \Diamond\boxminus\Phi \to \blacksquare\Phi \to \Diamond\Phi \end{gathered}$$

and this completes the proof. $\Box$

LEMMA 9
L is determined by

$$\begin{bmatrix} \forall u, v, w\ R(u,v) \wedge R(v,w) \to R(u,w) \\ \forall u, v\ R(u,v) \vee u = v \vee R(v,u) \\ \forall u\ \exists v\ R(u,v) \wedge \forall w\ R(u,w) \to R(v,w) \end{bmatrix}$$

Proof: As we know already the two axioms $\Phi \to \Box\diamondminus\Phi$ and $\Phi \to \boxminus\Diamond\Phi$ ensure that $R^{\smile}$ (the accessibility relation which belongs to $\boxminus$) is just the converse of R. Right-linearity is also ensured by $\diamondminus\Diamond\Phi \to (\diamondminus\Phi \vee \Phi \vee \Diamond\Phi)$. Remains to have a look at

$$\begin{array}{ccc} \blacklozenge\Phi & \to & \Box\diamondminus\Phi \\ \diamondminus\Phi & \to & \blacklozenge\Phi \\ \blacksquare\Phi & \to & \Diamond\Phi \end{array}$$

Applying the Elimination Theorem we get

$$\begin{gathered} \forall u, v, w\ S(u,v) \wedge R(u,w) \to R(v,w) \\ \forall u, v\ R(v,u) \to S(v,u) \\ \forall u\ \exists v\ S(u,v) \wedge R(u,v) \end{gathered}$$

All axioms involved are Sahlqvist formulas and so we know that L^{ex} is determined by

$$\alpha = \begin{bmatrix} \forall u, v\ R^{\smile}(u,v) \leftrightarrow R(v,u) \\ \forall u, v, w\ S(u,v) \wedge R(u,w) \to R(v,w) \\ \forall u, v\ R(v,u) \to S(v,u) \\ \forall u\ \exists v\ S(u,v) \wedge R(u,v) \\ \forall u, v, w\ R(u,v) \wedge R(u,w) \to R(v,w) \vee v = w \vee R(w,v) \end{bmatrix}$$

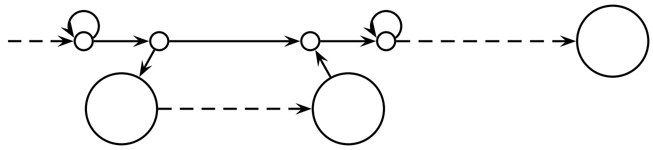
With the help of the Elimination Theorem we now have to compute $\exists^L \alpha = \exists S\ \exists R^{\smile}\ \alpha$ which – together with the generated model assumption results in what has been claimed. $\Box$

Thus transitivity and linearity are not affected by this approach. It is the density property which gets strengthened. In fact, the newly obtained property is a bit surprising at the first glance. One strange thing about it is that it implies the existence of accessible reflexive worlds which act as something like *next* worlds. The reason for this can be found in the rather loose definition of *density*. According to this definition even the structure $(\mathbb{N}, \leq)$ is dense simply because it is reflexive. From the achieved result we are ensured that under seriality, transitivity and linearity modal logics are not able to distinguish weak dense structures from *quasi-discrete* ones where quasi-discreteness is defined by

$$\forall u\ \exists v\ R(u,v) \wedge \forall w\ R(u,w) \to R(v,w)$$

It is evident that quasi-discreteness implies weak density and therefore in *weak dense* $KD4.3$ structures we may assume for every world that it is either reflexive or it is immediately followed by a cluster. Hence a typical weak dense structure looks like this:

$KD4.3$ with quasi-discreteness



Note that *weak dense* $KD4.3$ extends $KD4.2$ and therefore we may assume the existence of a final (non-degenerated) cluster.

COROLLARY 4
Quasi-discreteness may consistently be assumed for any logic extending weak dense $KD4.3$.

As another side-effect consider $(\mathbb{Q}, <)$, the structure we get from the rational numbers with the usual $<$ comparison. Evidently, $<$ is serial, transitive, linear and dense but as we just found out modal logics are not able to tell this structure from one we obtain if we additionally assume arbitrary intermediate clusters and so we get

COROLLARY 5
$(\mathbb{Q}, <)$ *is not modally axiomatizable.*

3.3.4 Logics extending $KD4.3$

For *weak dense* $KD4.3$ we reformulated both the density axiom and the transitivity axiom and finally ended up with a stronger property for density. There is no reason why we should not try the same for transitivity alone and see what we can get out of that then. In this case we consider

$$L = K(\Box) + \left\{ \begin{array}{c} \Box\Phi \to \Diamond\Phi \\ \Box\Phi \to \Box\Box\Phi \\ \Diamond\Phi \wedge \Diamond\Psi \to \Diamond(\Phi \wedge \Diamond\Psi) \vee \Diamond(\Phi \wedge \Psi) \vee \Diamond(\Diamond\Phi \wedge \Psi) \end{array} \right\}$$

and

$$L^{\mathrm{ex}} = K(\Box) + K(\boxminus) + K(\blacksquare) + \left\{ \begin{array}{c} \Phi \to \Box\diamondminus\Phi \\ \Phi \to \boxminus\Diamond\Phi \\ \blacksquare\Phi \leftrightarrow \Diamond\boxminus\Phi \\ \diamondminus\Phi \to \blacklozenge\Phi \\ \diamondminus\Diamond\Phi \to (\diamondminus\Phi \vee \Phi \vee \Diamond\Phi) \end{array} \right\}$$

and similarly to the former case we find out that L^{ex} is a normal conservative extension of L. Now L^{ex} is determined by

$$\alpha = \left[\begin{array}{l} \forall u, v\ R^{\smile} \leftrightarrow R(v,u) \\ \forall u, v, w\ S(u,v) \wedge R(u,w) \rightarrow R(v,w) \\ \forall u\ \exists v\ R(u,v) \wedge \forall w\ R(w,v) \rightarrow S(u,w) \\ \forall u, v\ R(u,v) \rightarrow S(v,u) \\ \forall u, v, w\ R(u,v) \wedge R(u,w) \rightarrow R(v,w) \vee v = w \vee R(w,v) \end{array}\right]$$

and hence L is determined by $\exists^L \alpha$, which is

$$\left[\begin{array}{l} \forall u, v, w\ R(u,v) \wedge R(v,w) \rightarrow R(u,w) \\ \forall u\ \exists v\ R(u,v) \wedge \forall w\ R(w,v) \rightarrow \forall w'\ R(u,w') \rightarrow R(w,w') \\ \forall u, v, w\ R(u,v) \wedge R(u,w) \rightarrow R(v,w) \vee v = w \vee R(w,v) \end{array}\right]$$

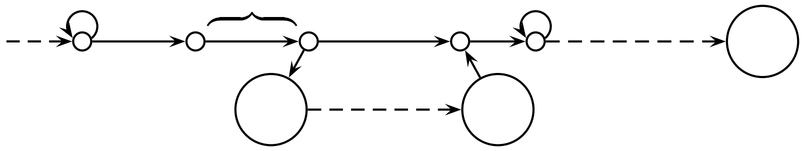
Evidently, under the generated model assumption we get linearity from right-linearity again, and so finally:

Lemma 10
*KD*4.3 *is determined by*

$$\left[\begin{array}{l} \forall u, v, w\ R(u,v) \wedge R(v,w) \rightarrow R(u,w) \\ \forall u\ \exists v\ R(u,v) \wedge \forall w\ R(w,v) \rightarrow \forall w'\ R(u,w') \rightarrow R(w,w') \\ \forall v, w\ R(v,w) \vee v = w \vee R(w,v) \end{array}\right]$$

Notice that the extra property we have got here is somewhat related to the property we got for dense structures. As a matter of fact it is trivially true for reflexive worlds (just take $v = u$) and also for worlds which have an *irreflexive next* world (let v be this next world then). Only in cases where a world is neither reflexive nor has an irreflexive next world this property tells us something new, namely that there exists a next cluster (a property we call *weak quasi-discreteness*). Thus a typical *KD*4.3 structure can be illustrated as follows (recall that we may assume the existence of a final cluster here as well):

*KD*4.3 with weak quasi-discreteness



Observe the little difference to the former figure for weak dense *KD*4.3. There the situation emphasized by the curly bracket would not be allowed.

3.4 Towards a Generalization

One of the main difficulties we are faced with when we want to apply the technique proposed is that the requirement of the new modality to be normal often induces other properties which may be undesired. As an example consider the logic $KT2$ (or $KD2$) which is like $S4.2$ but without the transitivity axiom $\Box\Phi \rightarrow \Box\Box\Phi$. Unfortunately, choosing again $\Diamond\Box$ as an auxiliary modality doesn't help here for $\Diamond\Box$ is not normal w.r.t. $KT2$. Transitivity would suffice as an additional property but what should we do if this extra property is to be avoided?

One possible solution to this problem can be found in an appropriate weakening of the requirements we had up to now. Although we know that $\Diamond\Box$ is not normal w.r.t. $KT2$ we nevertheless know (or at least can easily find out) that the rule

$$\vdash \Phi \rightarrow \Psi \Rightarrow\ \vdash \Diamond\Box\Phi \rightarrow \Diamond\Box\Psi$$

is valid. The reader familiar with minimal models and neighborhood semantics (see, e.g., [1]) will immediately recognize that this rule is associated with the so-called *strong neighborhood semantics*. This kind of semantics is slightly more general than the relational Kripke-semantics and defines $\Box\Phi$ to be true at world u if there exists a *neighborhood* for u whose elements all are Φ-worlds. We are thus talking about neighborhoods instead of accessibility relations. This strong neighborhood semantics is indeed more general than the Kripke-semantics as can be seen by the fact that under the assumption that the intersection of all neighbors of an arbitrary world u is itself a neighbor of u we may switch from neighborhoods to accessibility relations.

After we know that the chosen modality obeys the above rule there is no difficulty in finding a conservative extension[20] of the given logic: we just have to define a new modal operator in terms of this modality and show that no more theorems of the language we are interested in get provable this way. The actual proof is omitted here for it would be almost identical to the proof of Lemma 3.

What makes this generalization particularly interesting is the following.

LEMMA 11
Any modality $\mathbf{M}$ *made up of an arbitrary number of* $\Box$*'s and* $\Diamond$*'s obeys the inference rule schema*

$$\vdash \Phi \rightarrow \Psi \Rightarrow\ \vdash \mathbf{M}\,\Phi \rightarrow \mathbf{M}\,\Psi$$

Proof:

(1)	$\vdash \Phi \rightarrow \Psi$	given
(2)	$\vdash \Box(\Phi \rightarrow \Psi)$	from (1) and necessitation rule
(3)	$\vdash \Box(\Phi \rightarrow \Psi) \rightarrow (\Box\Phi \rightarrow \Box\Psi)$	K-Axiom
(4)	$\vdash \Box\Phi \rightarrow \Box\Psi$	Modus Ponens (2) and (3)

(1)	$\vdash \Phi \rightarrow \Psi$	given
(2)	$\vdash \neg\Psi \rightarrow \neg\Phi$	classical propositional calculus
(3)	$\vdash \Box\neg\Psi \rightarrow \Box\neg\Phi$	as above
(4)	$\vdash \neg\Box\neg\Phi \rightarrow \neg\Box\neg\Psi$	classical propositional calculus
(5)	$\vdash \Diamond\Phi \rightarrow \Diamond\Psi$	definition of $\Diamond$

The claim of the lemma then follows by a trivial induction on the length of the modality. $\Box$

[20]The newly to be defined modal operator won't be normal; therefore we are not talking of *normal* conservative extensions anymore.

Coming back to the example $KT2$ we therefore consider the logics

$$L = K(\Box) + \left\{ \begin{array}{c} \Box\Phi \to \Phi \\ \Diamond\Box\Phi \to \Box\Diamond\Phi \end{array} \right\}$$

$$L^{\text{ex}} = K(\Box) + E(\blacksquare) + \left\{ \begin{array}{c} \Box\Phi \to \Phi \\ \blacksquare\Phi \to \blacklozenge\Phi \\ \Diamond\Box\Phi \to \blacksquare\Phi \end{array} \right\}$$

where

$$E(\blacksquare) = \{\vdash \Phi \to \Psi \Rightarrow \; \vdash \blacksquare\Phi \to \blacksquare\Psi\}$$

It is immediate that L^{ex} is a conservative extension of L, or in other words, that for every formula $A \in \mathcal{L}_L$: A is an L^{ex}-theorem if and only if A is an L-theorem. According to the strong neighborhood semantics we now translate modal formulas as

$$\lceil \blacksquare\Phi \rceil^u = \exists x \; N(u,x) \wedge \forall v \; I(x,v) \to \lceil \Phi \rceil^v$$

where the translation of the other operators and connectives remains as before. Then $\Diamond\Box\Phi \to \blacksquare\Phi$ corresponds to $\forall u, v \; R(u,v) \to \exists x \; N(u,x) \wedge \forall w \; I(x,w) \to R(v,w)$ and $\blacksquare\Phi \to \blacklozenge\Phi$ corresponds to $\forall x, u, y \; N(x,u) \wedge N(x,y) \to \exists z \; I(y,z) \wedge I(u,z)$ such that L is determined by

$$\exists I, N \left[\begin{array}{l} \forall u \; R(u,u) \\ \forall x, u, y \; N(x,u) \wedge N(x,y) \to \exists z \; I(y,z) \wedge I(u,z) \\ \forall u, v \; R(u,v) \to \exists x \; N(u,x) \wedge \forall w \; I(x,w) \to R(v,w) \end{array} \right]$$

By the Elimination Theorem this is equivalent to the conjunction of reflexivity and[21]

$$\exists f, g \left[\begin{array}{l} \forall z, u \; R(z,u) \to \forall v \; R(u, f(v, g(z,u))) \\ \forall x, y, z, u, v, w \; R(x,y) \wedge R(z,u) \wedge g(z,u) = g(x,w) \to R(u, f(g(z,u), g(x,v))) \end{array} \right]$$

It is not at all obvious whether this formula is really equivalent to a first-order formula. However, it evidently implies

$$\exists f, g \left[\begin{array}{l} \forall z, u \; R(z,u) \to \forall v \; R(u, f(v, g(z,u))) \\ \forall z, u, v \; R(z,u) \to R(u, f(g(z,u), g(z,v))) \end{array} \right]$$

and therefore also implies

$$\exists h \left[\begin{array}{l} \forall z, u \; R(z,u) \to \forall v \; R(u, h(z,v,u)) \\ \forall z, u \; R(z,u) \to \forall v \; R(u, h(z,u,v)) \end{array} \right]$$

This latter formula can be deskolemized such that we finally obtain

$$\forall z, u, v \, \exists w \; (\neg R(z,v) \vee R(v,w)) \wedge (\neg R(z,u) \vee R(u,w))$$

Interestingly, this result is logically equivalent to *directedness*. In a sense we therefore havn't gained anything by applying the technique proposed in this paper. However,

[21] The reader who is a bit confused about the kind of skolemization used here is encouraged to have a look at [8].

in theorem proving on the basis of translating modal logic formulas into first-order logic it might make sense to prefer the newly found formula. This holds in particular for resolution-based theorem provers which in general will prefer simple two-literal clauses over three-literal clauses.

An even stronger generalization can be found in considering the usual (weak) neighborhood semantics in which a formula like $\Box\Phi$ is to be translated as

$$\lceil \Box\Phi \rceil^u = \exists x\ N(u,x) \wedge \forall v\ (I(x,v) \leftrightarrow \Phi(v))$$

Dealing with this translation (semantics) is even more complicated. However, it is indeed much more general for the only rule (no axioms) which is induced by this semantics is

$$N(\Box) \quad = \quad \vdash \Phi \leftrightarrow \Psi \Rightarrow\ \vdash \Box\Phi \leftrightarrow \Box\Psi$$

With this we may define new modal operators more or less arbitrarily (not necessarily by modalities) and still remain conservative.

Lemma 12
Let $\Delta(P)$ be a modal formula with P as its only propositional variable. If $\models \Phi \Leftrightarrow \Psi$ then $\models \Delta[P \leftarrow \Phi] \Leftrightarrow \Delta[P \leftarrow \Psi]$.
Proof: Simple induction on the structure of Δ. $\Box$

For example, consider the definition $\blacksquare\Phi \leftrightarrow (\neg\Phi \wedge \Box\Phi)$. Under this definition alone we have neither $K(\blacksquare)$ nor $E(\blacksquare)$ but we certainly do have $N(\blacksquare)$. Then consider the modal logic

$$L = \text{Löb} = K(\Box) + \{\Box(\Box\Phi \to \Phi) \to \Box\Phi\}$$

and its conservative extension

$$L^{\text{ex}} = K(\Box) + N(\blacksquare) + \left\{ \begin{array}{c} \Box\blacklozenge\Phi \to \Box\Phi \\ (\Box\Phi \to \Phi) \to \blacklozenge\Phi \end{array} \right\}$$

The second of the two new axioms causes no real problems. The first, however, cannot be transformed into a first-order property by the Elimination Theorem; it results in a formula with a second-order skolem function which apparently cannot be deskolemized. Therefore the α for L^{ex} is second-order and the $\exists^L\alpha$ for L remains second-order and we haven't really gained anything.

Because of its generality, the neighborhood semantics leads quite often to such problems. It is not entirely clear yet whether this is due to a weakness in the elimination methods we are using or whether the formulas we obtain on the basis of the neighborhood semantics are inherently second-order. In the first case stronger and more general elimination techniques might help whereas in the latter case we are forced to look for suitable approximations.

4 Summary, Conclusion and Further Work

Finding out about determining properties for modal logics is interesting in at least two respects. From a model-theoretic perspective they tell us something about the models underneath these logics and also about the (limits of the) expressive power of modal logics. From a proof-theoretic point of view it is of quite some importance to find characterizations which are "as strong as possible" because predicate logic theorem provers will usually have less difficulties then in proving the validity of theorems under semantics-based translations. It might even be possible that such a strengthening can influence other kind of calculi like tableaux for instance.

The approach presented in this paper is based on the observation that it is often easy to find a determining property for some conservative extension of the logic under consideration which then can be used to extract a stronger determining property of this original logic. Such conservative extensions are constructed in a more or less straightforward manner. Essentially we try to find a suitable auxiliary modality which defines a new modal operator. This is the only "creative" part of the method proposed. What remains to be done consists merely of several applications of a second-order quantifier elimination and we are done.

Evidently, there are various possibilities where this approach may not work or does not really help. What may go awry is the attempt to find normal conservative extensions. In this case it is certainly worth having a look at the possible generalizations as they are described in Section 3.4. But even if we are able to find a normal conservative extension it is not yet guaranteed that we can find a determining property for this one. Usually, however, both possible obstacles occur rarely, and if we are successful up to this stage the most important steps have been performed. All the rest can be done via the Elimination Theorem which either results in the original determining property we already knew about (and this is the worst case to happen for we didn't gain anything then) or we end up with a first-order property which is strictly stronger than what we had before (the best we can achieve) or, a final possibility, we get an existentially quantified second-order formula which is strictly stronger than the original one. In the latter case the result obtained can still be used in refutation procedures although it certainly requires a careful examination whether this stronger property really simplifies the derivation process.

What is interesting about the approach presented in this paper are not only the results obtained; also the way how this is done is of some importance. Whereas other techniques with a similar goal are usually based on pure model-theoretic considerations, the approach presented here is mainly proof-theoretic and, in the author's view, requires much less "intuition" on the underlying models and frames than other known techniques (as e.g. Lemmon-filtrations).

Nevertheless, there are some severe limitations in applying the approach. These are mainly due to the (still) rather weak techniques to eliminate second-order quantifiers. Putting it to the extreme: Finding conservative extensions is actually no problem at all if we are prepared to consider neighborhood semantics as the basis for the semantical definition of the auxiliary modal operators. The price we have to pay for this is that it is often very hard to get first-order correspondence results out of this. Of course, it is not always the case that there are such first-order correspondence results at all; it is just that we want to be able to find out if there are. And the stronger the elimination procedure is, the better results can be expected.

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